SOLUTIONS OF RELATION, FUNCTION & ITF

EXERCISE # 1

PART–1

Section (A)

A.1 n(A × B) = n(A) × n(B) = 3 × 3 = 9

A.2 A = {2, 3}, B = {2, 4}, C = {4, 5}
$B \cap C = \{4\}$ ∴ $A \times (B \cap C) = \{(2, 4), (3, 4)\}$.

A.3 Number of relation from A to B = 2^{12}

A.4 If x = 2 then y = 1, If x = 3 then y = 3,
If x = 4 then y = 5, If x = 5 then y = 7,

A.5 **(i)** Domain of R = first element of pairs (x, y) = {– 3, – 2, – 1, 0, 1, 2, 3}
(ii) Range set consisting of 4, 3, 2, 1, 0, 1, 2 = {0, 1, 2, 3, 4}
(iii) $-3 \le x \le 3 \Rightarrow (x, y) = (x, \sqrt{(x-1)^2}\,) = \{(-3, 4), (-2, 3), (-1, 2), (0, 1), (1, 0), (2, 1), (3, 2)\}$

A.6 $(-1,2) \in A \times A$
⇒ $-1 \in A$, $2 \in A$ and $(0, 1) \in A \times A \Rightarrow 0 \in A$, $1 \in A$
So A = {–1, 0, 1, 2} as A has four elements and S = {–1, 0), (1,–1), (–1, 2), (0, 1), (0, 2), (1, 2)}
Hence the required element of S are given by (a)

Section (B)

B.1 **(i)** For Reflexive
$R \in \{(1, 1)\ (2, 2)\ (3, 3)\}$
For symmetric $(1, 2) \in R$ but $(2, 1) \notin R$ Not symmetric
for transitive $(1, 2), (2, 3) \in R \Rightarrow (1, 3) \in R$ so transitive
(ii) Obviously, the relation P is neither reflexive nor transitive but it is symmetric, because $x_2 + y_2 = 1 \Rightarrow y_2 + x_2 = 1$.

B.2 $x < y,\ y < z \Rightarrow x < z\ \forall\ x, y, z \in N$
∴ xRy yRz ⇒ xRz,
∴ Relation is transitive,
∴ x < y does not give y < x.
Relation is not symmetric. Since x < x does not hold, hence relation is not reflexive.

B.3 (i) $(q, q) \notin R$ — Not reflexive; (ii) $(p, q) \notin R$ — not symmetric; (iv) $(q, p) \notin R$ — not symmetric

B.4 $2x + x = 41 \Rightarrow x = \frac{41}{3} \notin N$ ∴ R is not reflexive
$2x + y = 41 \not\Rightarrow 2y + x = 41$ ∴ R is not symmetric
$2x + y = 41$ and $2y + z = 41 \Rightarrow 4x - z = 41 \Rightarrow (x, z)\ R$
∴ R is not transitive

B.5 $aRb \Leftrightarrow n|(a-b)$ $a, b \in Z$
$n \in I^+$
(i) $aRa \Leftrightarrow n|(a-a) \Leftrightarrow$ so R is reflexive
(ii) $aRa \Leftrightarrow n|(a-b) = n|(b-a) \Leftrightarrow$ R is symmetric
(iii) $aRb \Leftrightarrow n|(a-b)$ and $n|(b-c)$
$\Rightarrow n|(a-b) + (b-c) \Leftrightarrow n|(a-c) \Leftrightarrow$ R is transitive

B-6. Reflexive Relation :
A.A = A.A for $\forall A \in S$, so Relation is Reflexive Relation
Symmetric Relation : A.B = BA ⇒ BA = AB ∀ A, B∈S, so Relation is Symmetric Relation
Transitive Relation : AB = BA, BC = CB ⇒ AC = CA Not True, ∀ A, B, C∈S

for example $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \Rightarrow AB = BA, BC = CB$ but $AC \neq CA$

so Relation is not Transitive Relation

Section (C)

C-1. **(i)** **Case-I :** Let $x = I$

$2I = I + 0 \Rightarrow I = 0$

Case-II : Let $x = I + f, f \in (0, 1)$

given $2I = I + f + f \Rightarrow f = \frac{I}{2}$

Hence $I = 1$ and $f = \frac{1}{2}$

so $x = 0, \frac{3}{2}$

C-2 **(i)** **Case - I :** $x \in I$ then $4x = x$

$x = 0.$

Case-II : $x \notin I$, then $4[x] = [x] + 2\{x\} \Rightarrow \{x\} = \frac{3[x]}{2}$

$0 \leq \{x\} < 1 \Rightarrow 0 \leq [x] < \frac{2}{3}$

$\therefore [x] = 0 \quad \therefore \{x\} = 0 \quad \therefore x = 0 + 0 = 0$

(ii) $[x]^2 = -[x]$

$x = I + f$

$I^2 = -I$

$I = 0$ or $I = -1$

Case-I $I = 0$

$x = I + f = f$

$x \in [0, 1)$ ....(i)

Case-II $I = -1$

$-1 \leq x < 0, x \in [-1, 0)$ ....(ii)

by (i) and (ii)

$x \in [-1, 1)$

(iii) $\{x\} = 0$ or $\{x\} = -1$

$x \in I$ (rejected)

(iv) Let $x \in I + f_1$ $f \in [0, 1)$

Case : $f \in \left[0, \frac{1}{2}\right)$ $2I = I \Rightarrow I = 0 \Rightarrow x \in \left[0, \frac{1}{2}\right)$ Hence: $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$

C-3. **(i)** $-5 \leq [x + 1] < 2$

$-5 \leq [x + 1] \leq 1$

$\Rightarrow -5 \leq x + 1 < 2$

$-6 \leq x < 1 \Rightarrow x \in [-6, 1)$

(ii) $[x]^2 + 5[x] - 6 < 0 \Rightarrow ([x] + 6)([x] - 1) < 0 \Rightarrow -6 < [x] < 1$

$\Rightarrow -5 \leq [x] \leq 0 \Rightarrow -5 \leq x < 1 \Rightarrow x \in [-5, 1)$

(iii) $-1 < \{x\} < 0$ is not possible

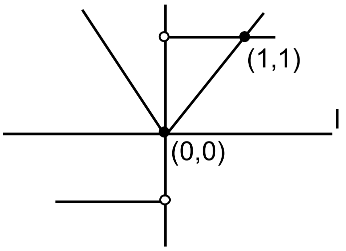
hence $x \in \phi$

(iv) $-1 \leq [x] \leq 0 \Rightarrow -1 \leq x < 1.$

C-4. **(i)** $\{[x]\} = 0 \Rightarrow [x] \in I \Rightarrow x \in R$

(ii) $x^2 - 2x - 8 < 0 \Rightarrow x \in (-2,4)$

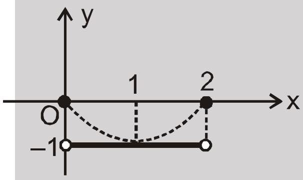
C-5. **(i)**



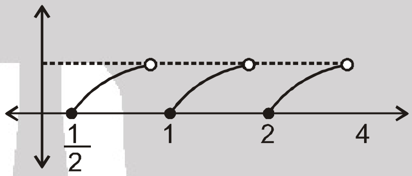
$\Rightarrow$ two solution

(ii) Case-I $(x + 1)^2 = 0$ & sgn $(x^2 - 1) = 0 \Rightarrow x = -1$
Case-II $(x + 1)^2 = 1$ & sgn $(x^2 - 1) = 1 \Rightarrow x = -2$
$\Rightarrow x \in \{-1, -2\}$

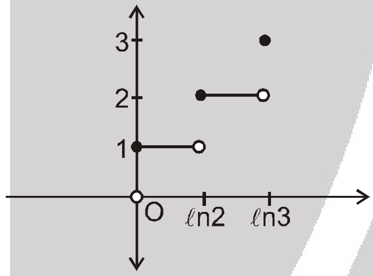
C-6. **(i)**



(ii)



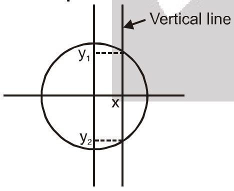
(iii)



C-7. $f(0) = 0$ as $0 \in Q$
$f(e) = 1 - e$ as $e \in Q^c$
$[f(e)] = -2 \qquad \Rightarrow \qquad |\,[f(e)]\,| = 2$
Hence sum is 3

Section (D)

D-1. $x^2 + f(x)^2 = 36 \quad \Rightarrow \quad f(x) = \pm\sqrt{36 - x^2}$
Equation represents circle of radius 6 with centre (0, 0)
By vertical line test, for every value of x, there are two values of y .
Which contradicts definition of function
So equation doesn't represents a function.



D-2. **(i)** $f(x) = \dfrac{x^3 - 5x + 3}{x^2 - 1} \Rightarrow f(x) = \dfrac{x^3 - 5x + 3}{(x-1)(x+1)}$. Division by zero is undefined
$\therefore \quad x \neq \pm 1$
$\therefore \quad$ Domain $x \in R - \{1, -1\} \Rightarrow x \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

(ii) $\cos x \in [0, 1] \Rightarrow 2n\pi - \dfrac{\pi}{2} \le x \le 2n\pi + \dfrac{\pi}{2}$, $n \in I$

(iii) $f(x) = \dfrac{1}{\sqrt{x + |x|}}$ for function to be defined $x + |x| > 0$ for $x > 0$, $\quad x + |x| = 2x > 0$
for $x \le 0$, $x + |x| = 0$
$\therefore \quad$ Domain is $x \in (0, \infty)$

(iv) $f(x) = e^{x + \sin x}$. Domain $x \in R$ as there is no restriction for exponent of e.

(v) $f(x) = \dfrac{1}{\log_{10}(1-x)} + \sqrt{x+2}$

$1 - x > 0$ and $x + 2 \ge 0$ and $1 - x \ne 1 \Rightarrow x \in (-\infty, 1) - \{0\}$ and $x \ge -2 \Rightarrow x \in [-2, 0) \cup (0, 1)$

(vi) Clearly $x > 2$ and $\dfrac{\log_2(x-2)}{\log_{1/2}(3x-1)} \ge 0 \Rightarrow \log_2(x-2) \le 0 \Rightarrow x - 2 \le 1 \Rightarrow x \le 3$

(vii) $x^2 + x + 1 \ge 1 \quad \Rightarrow \quad (-\infty, -1] \cup [0, \infty)$

(viii) $f(x) = \dfrac{\sqrt{\cos x - \frac{1}{2}}}{\sqrt{6+35x-6x^2}} \Rightarrow \cos x - \dfrac{1}{2} \ge 0$ or $x \in \left[2n\pi - \dfrac{\pi}{3}, 2n\pi + \dfrac{\pi}{3}\right]$, $n \in I$

and $6 + 35x - 6x^2 > 0$ or $x \in \left(-\dfrac{1}{6}, 6\right) \Rightarrow$ Domain $\left(-\dfrac{1}{6}, \dfrac{\pi}{3}\right] \cup \left[\dfrac{5\pi}{3}, 6\right)$

D-3. **(i)** $f(x) = \sqrt{3 - 2^x - 2.2^{-x}}$

$3 - 2^x - 2\,.2^{-x} \ge 0$

or $\quad (2^x)^2 - 3.2^x + 2 \le 0$

or $\quad (2^x - 1)(2^x - 2) \le 0$

$\Rightarrow \quad 2^x \in [1, 2] \quad \Rightarrow \quad x \in [0, 1]$

(ii) $f(x) = \sqrt{1 - \sqrt{1-x^2}}$

$1 - \sqrt{1-x^2} \ge 0 \quad \Rightarrow \quad \sqrt{1-x^2} \le 1$

$\Rightarrow \; 0 \le 1 - x^2 \le 1 \quad \Rightarrow \quad x \in [-1, 1]$

(iii) $f(x) = (x^2 + x + 1)^{-3/2} \quad \Rightarrow \quad D : x \in R$

(iv) $f(x) = \sqrt{\dfrac{x-2}{x+2}} + \sqrt{\dfrac{1-x}{1+x}} \quad \Rightarrow \quad \dfrac{x-2}{x+2} \ge 0$ and $\dfrac{1-x}{1+x} \ge 0$

$x \in (-\infty, -2) \cup [2, \infty)$ and $x \in (-1, 1]$ $D : \phi$

(v) $f(x) = \sqrt{\tan x - \tan^2 x} \quad \Rightarrow \quad \tan x - \tan^2 x \ge 0$

or $\quad 0 \le \tan x \le 1 \quad$ or $\quad x \in \bigcup\limits_{n \in I} \left[n\pi, n\pi + \dfrac{\pi}{4}\right]$

(vi) $f(x) = \dfrac{1}{\left|2\sin\frac{x}{2}\right|} \quad \Rightarrow \quad \sin\dfrac{x}{2} \ne 0 \quad$ or $\quad x \ne 2n\pi$

(vii) $f(x) = \sqrt{\log_{1/4}\left(\dfrac{5x-x^2}{4}\right)} \quad \Rightarrow \quad \dfrac{5x-x^2}{4} \le 1 \quad$ and $\quad 5x - x^2 > 0$

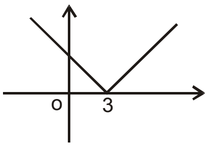
or $\quad x \in (0, 1] \cup [4, 5)$

(viii) $f(x) = \log_{10}(1 - \log_{10}(x^2 - 5x + 16)) \Rightarrow \quad 1 - \log_{10}(x^2 - 5x + 16) > 0$

or $\quad x^2 - 5x + 6 < 0 \quad$ or $\quad x \in (2, 3)$.

D-4. **(i)** $y = |x - 3|$

Range $y \in [0, \infty)$



(ii) $y = \dfrac{x}{1+x^2}$

Method 1

Domain $x \in R$

$yx^2 - x + y = 0$

quadratic in x has real roots as $x \in R$

$\therefore$ Discriminant $D \geq 0 \Rightarrow 1 - 4y^2 \geq 0 \Rightarrow (2y - 1)(2y + 1) \leq 0i \Rightarrow y \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

Here at y = 0 quadratic vanishes. so we have to check this seperately

Put $y = 0 \Rightarrow$ $x = 0$ (a point with in domain)

$\therefore$ $y = 0$ point is included in the range

Note : If there is no point of x in the domain for the value of y for which quadratic vanishes, we have to remove that point from range

Method 2

$f(x) = \dfrac{x}{1+x^2} = \dfrac{1}{\left(\dfrac{1}{x}+x\right)}$ We know that $\left|x + \dfrac{1}{x}\right| \geq 2$

$\Rightarrow 0 < \dfrac{1}{\left|x+\dfrac{1}{x}\right|} \leq \dfrac{1}{2} \Rightarrow \dfrac{1}{\left(x+\dfrac{1}{x}\right)} \in \left[-\dfrac{1}{2}, 0\right) \cup \left(0, \dfrac{1}{2}\right]$ But divison by x is done by us,

So at x = 0, y = 0

$\therefore$ Range $y \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

Method 3

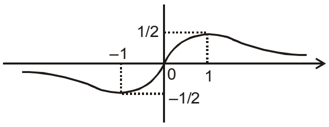
$f(x) = \dfrac{x}{1+x^2}$ is an odd function $\Rightarrow f'(x) = \dfrac{1-x^2}{(1+x^2)^2} = 0 \quad x = \pm 1$

$> 0 \quad x \in (-1, 1)$

$< 0 \quad x \in (-\infty, -1) \cup (1, \infty)$

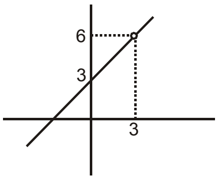
$\lim\limits_{x\to\infty} \dfrac{x}{1+x^2} = 0$ (0^+ more accurately)

$\lim\limits_{x\to-\infty} \dfrac{x}{1+x^2} = 0$ (0^- more accurately)



Range $y \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

(iii) $f(x) = \dfrac{x^2-9}{x-3} = \dfrac{(x-3)(x+3)}{(x-3)} = (x+3)$



Domain $x \in R - \{3\}$

Range $y \in R - \{6\}$

(iv) $f(x) = \sin^2(x^3) + \cos^2(x^3)$

$f(x) = 1$

Domain $x \in R$

Range $y \in \{1\}$

(v) $f(x) = 3 \sin x + 4 \cos x + 5$

$-\sqrt{3^2+4^2} \le 3 \sin x + 4 \cos x \le \sqrt{3^2+4^2} \Rightarrow -5 \le 3 \sin x + 4 \cos x \le 5$

$\Rightarrow 0 \le 3 \sin x + 4 \cos x + 5 \le 10 \Rightarrow$ Range $y \in [0, 10]$

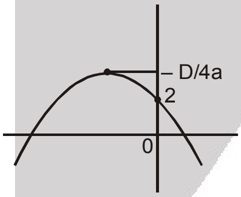
(vi) $f(x) = 2 - 3x - 5x^2$

Domain $x \in R$

Method 1

$y = -5x^2 - 3x + 2$

opening downward parabola



Range $y \in \left(-\infty, \dfrac{-D}{4a}\right] \Rightarrow y \in \left(-\infty, \dfrac{49}{20}\right]$

Method 2

$5x^2 + 3x + (y-2) = 0$

$D \ge 0 \Rightarrow 9 - 20(y-2) \ge 0 \Rightarrow 20y - 49 \le 0 \Rightarrow y \le \dfrac{49}{20}$

(vii) $f(x) = \dfrac{x+2}{x^2-8x-4} = y \Rightarrow x + 2 = yx^2 - 8yx - 4y$

or $yx^2 - x(8y+1) - (4y+2) = 0$

for x to be real $D \ge 0$

$(8y+1)^2 + 4y(4y+2) \ge 0 \Rightarrow 64y^2 + 16y + 1 + 16y^2 + 8y \ge 0$

$80y^2 + 24y + 1 \ge 0$ or $y \in \left(-\infty, -\dfrac{1}{4}\right] \cup \left[-\dfrac{1}{20}, \infty\right)$

(viii) $f(x) = \dfrac{x^2-2x+4}{x^2+2x+4} = y \Rightarrow x^2 - 2x + 4 = yx^2 + 2xy + 4y$

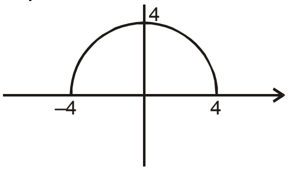
$x^2(1-y) - 2x(1+y) + 4(1-y) = 0$

$D \ge 0$

$4(1+y)^2 - 16(1-y)^2 \ge 0$ or $y \in \left[\dfrac{1}{3}, 3\right]$

D-5. **(i)** $f(x) = \sqrt{16-x^2}$. Domain $x \in [-4, 4] \Rightarrow f(x) > 0,\ y = \sqrt{16-x^2} \Rightarrow x^2 + y^2 = 16$

Equation of semicircle



$\therefore$ Range $y \in [0, 4]$

(ii) $f(x) = \dfrac{1}{\sqrt{4+3\sin x}}$

Domain $4 + 3 \sin x > 0 \Rightarrow \sin x > -\dfrac{4}{3}$ Always true $\Rightarrow x \in R$

Range $-3 \le 3 \sin x \le 3$

$\Rightarrow 1 \le 4 + 3 \sin x \ge 7 \Rightarrow 1 \ge \dfrac{1}{4+3\sin x} \ge \dfrac{1}{7} \Rightarrow 1 \ge \dfrac{1}{\sqrt{4+3\sin x}} \ge \dfrac{1}{\sqrt{7}} \Rightarrow y \in \left[\dfrac{1}{\sqrt{7}}, 1\right]$

(iii) $f(x) = \dfrac{1}{1+\sqrt{x}}$; $\infty > \sqrt{x} \ge 0 \Rightarrow \infty > 1 + \sqrt{x} \ge 1 \Rightarrow 0 < \dfrac{1}{1+\sqrt{x}} \le 1$

$\therefore$ Range $y \in (0, 1]$

(iv) $f(x) = \ell n\left(\dfrac{\sqrt{8-x^2}}{x-2}\right)$ for f' to be defined $8 - x^2 > 0$ & $\dfrac{\sqrt{8-x^2}}{x-2} > 0$; $x > 2$

$(2, 2, \sqrt{2})$ $\Rightarrow$ $0 < \dfrac{\sqrt{8-x^2}}{x-2} < \infty\ \forall\ x \in (2, 2\sqrt{2})$

Range of $\ell n\dfrac{\sqrt{8-x^2}}{x-2} \in (-\infty, \infty) = R$

(v) $f(x) = \left[\dfrac{1}{\sin\{x\}}\right] = \Rightarrow \sin\{x\} \ne 0 \Rightarrow \{x\} \ne n\pi,\ n \in I$

$0 < \sin\{x\} < \sin 1 < \sin 60° \Rightarrow 0 < \sin\{x\} < \dfrac{\sqrt{3}}{2} \Rightarrow \dfrac{2}{\sqrt{3}} < \dfrac{1}{\sin\{x\}} < \infty$

$\left[\dfrac{1}{\sin\{x\}}\right] = 1, 2, 3\ldots\ldots\ldots \Rightarrow$ Range of $\left[\dfrac{1}{\sin\{x\}}\right] \in N$

(vi) $f(x) = \dfrac{1}{\sqrt{16-4^{x^2-x}}}$ for function to be defined

$16 - 4^{x^2-x} > 0 \Rightarrow 16 > 4^{x^2-x} \Rightarrow 4^{x^1-x} < 4^2$

$\Rightarrow x^2 - x - 2 < 0 \Rightarrow (x-2)(x+1) < 0$

so $n \in (-1, 2) \Rightarrow$ " $x \in (-1, 2) \Rightarrow x^2 - x \in \left[-\dfrac{1}{4}, 2\right]$

$4^{x^1-x} \in \left[\dfrac{1}{\sqrt{2}}, 16\right) \Rightarrow \sqrt{16-4^{x^2-x}} \in \left(0, \sqrt{16-\dfrac{1}{\sqrt{2}}}\right]$

So range of $\dfrac{1}{\sqrt{16-4^{x^2-x}}} \in \left[\dfrac{1}{\sqrt{16-\dfrac{1}{\sqrt{2}}}}, \infty\right]$

(vii) $f(x) = \dfrac{1}{2-\cos 3x}$ range of $\cos 3x$ is $[-1, 1]$ $\cos 3x\ [-1, 1]$

$\therefore$ $f(x) \in \left[\dfrac{1}{3}, 1\right]$

(viii) $f(x) = 3\sin\sqrt{\dfrac{\pi^2}{16} - x^2}$ $\Rightarrow$ $D : x \in \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$ $\Rightarrow$ $\sqrt{\dfrac{\pi^2}{16} - x^2} \in \left[0, \dfrac{\pi}{4}\right]$

$\therefore$ $f(x) \in \left[0, \dfrac{3}{\sqrt{2}}\right]$

(ix) $f(x) = \sin^2 x + \cos^4 x = \sin^2 x + 1 + \sin^4 x - 2\sin^2 x$ $= \sin^4 x - \sin^2 x + 1 = \left(\sin^2 x - \dfrac{1}{2}\right) + \dfrac{3}{4}$

$R : \left[\dfrac{3}{4}, 1\right]$.

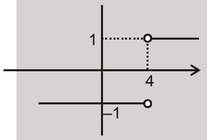
(x) Domain is $R - (2n + 1)\ \dfrac{\pi}{2}$ and $-\sqrt{2} \le \sin x + \cos x \le \sqrt{2}$. $\sin(\sin x + \cos x) \ne \pm \sin 1$

But these values will come at $x = 0, \pi$ so cannot be excluded.

$R - (2n + 1)\ \dfrac{\pi}{2}$ $-\sqrt{2} \le \sin x + \cos x \le \sqrt{2}$. $\sin x + \cos x \ne \pm \sin 1$

(xi) $f(x) = x^3 - 2x^2 + 5 = (x^2 - 1)^2 + 4$ $\Rightarrow$ $R : [4, \infty)$

D-6. **(i)** $f(x) = \dfrac{|x-4|}{x-4}$, $x \ne 4$



$f(x) = \begin{cases} 1 & , \ x > 4 \\ -1 & , \ x < 4 \end{cases}$

$\therefore$ Range $y \in \{-1, 1\}$

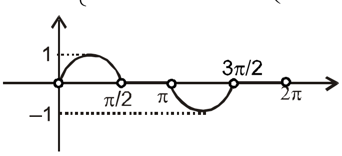
(ii) $f(x) = 3|\sin x| - 4|\cos x|$. $f(x)$ is a periodic function with period π. So analysis is limited in $[0, \pi]$

$f_{max} = 3.1 - 4.0 = +3$ at $x = \dfrac{\pi}{2}$, $|\sin x| = 1$, $|\cos x| = 0$

$f_{min} = 3.0 - 4.1 = -4$ at $x = 0$, $|\sin x| = 0$, $|\cos x| = 1$ $\therefore$ Range $y \in [-4, 3]$

(iii) $f(x) = \dfrac{\sin x}{\sqrt{1+\tan^2 x}} + \dfrac{\cos x}{\sqrt{1+\cot^2 x}}$. $f(x) = \sin x\,|\cos x| + \cos x\,|\sin x|$ periodic period $= 2\pi$

$$f(x) = \begin{cases} \sin 2x & , \ x \in \left(0, \dfrac{\pi}{2}\right) \\ 0 & , \ x \in \left(\dfrac{\pi}{2}, \pi\right) \\ -\sin 2x & , \ x \in \left(\pi, \dfrac{3\pi}{2}\right) \\ 0 & , \ x \in \left(\dfrac{3\pi}{2}, 2\pi\right) \end{cases}$$



$\therefore$ Range $y \in [-1, 1]$

(vi) $f(x) = 1 - |x - 2|$

$\Rightarrow$ $|x - 2| \in [0, \infty)$

$\therefore$ $f(x) \in (-\infty, 1]$

(v) $f(x) = x^3 - 12x\ ,\ x \in [-3, 1] = x\,(x^2 - 12)$

$\Rightarrow$ $f'(x) = 3x^2 - 12 = 0$

or $x = \pm 2$

$R : [-11, 16]$

(vi) $f(x) = [\sin x] + [\tan x] + [\cos x] + [\sec x]$

$\because$ $x \in (0, \pi/4)$

$\Rightarrow$ $\sin x \in \left(0, \dfrac{1}{\sqrt{2}}\right) \Rightarrow [\sin x] = 0$

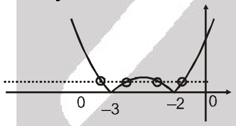
$\Rightarrow$ $\cos x \in \left(\dfrac{1}{\sqrt{2}}, 1\right) \Rightarrow [\cos x] = 0$

$\tan x \in (0, 1) \Rightarrow [\tan x] = 0$

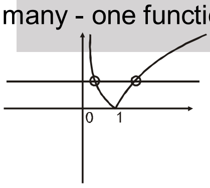
$\Rightarrow$ $\sec x \in (1, \sqrt{2}\ ) \Rightarrow [\sec x] = 1$

range of $f(x) = \{1\}$

D-7. **(i)** $y = |(x + 2)\,(x + 3)|$
many - one function



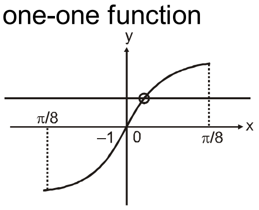
(ii) $y = |\ell nx|$
many - one function



(iii) $f(x) = \sin 4x,\ x \in \left(-\dfrac{\pi}{8}, \dfrac{\pi}{8}\right)$

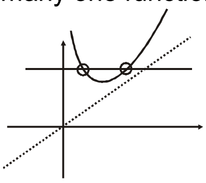
period $= \dfrac{\pi}{2}$

one-one function



(iv) $f(x) = x + \frac{1}{x}$, $\quad x\ \frac{1}{x} \in (0, \infty)$

many one function



(v) $f(x) = \sqrt{1-e^{\left(\frac{1}{x}-1\right)}} \Rightarrow f' = \dfrac{e^{\left(\frac{1}{x}-1\right)}\cdot\frac{1}{x^2}}{2\sqrt{1-e^{\left(\frac{1}{x}-1\right)}}} > 0$ increasing function

Hence one - one

(vi) $f(x) = \frac{3x^2}{4\pi} - \cos(\pi x)$ even function

Hence many - one

(vii) $f(x) = x^3 + \frac{1}{x^3} \Rightarrow f'(x) = 3x^2 - \frac{3}{x^4} = 0 \Rightarrow \quad x = \pm 1$

Also $\quad f(x) \neq 0$

$\therefore \quad$ Range $\notin$ R

$\therefore \quad$ f(x) is into function.

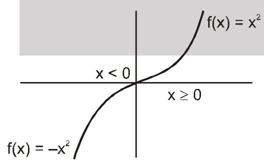
(viii) f(x) = x cos x odd function (f(0) = 0)

$\Rightarrow \quad f'(x) = \cos x - x \sin x$

f(x) is an odd continuous function for which $\left|\lim_{x\to\infty} x\cos x\right| = \infty$. Hence range $\in$ R onto function

(ix) $f(x) = \frac{1}{\sin\sqrt{|x|}}$ Clearly many one. Clearly $f(x) \in (-\infty, -1] \cup [1, \infty) \Rightarrow$Range $\notin$ R into function

D-8. **(i)** f(x) = x|x|



$= \begin{cases} x^2 & , \quad x \geq 0 \\ -x^2 & , \quad x < 0 \end{cases}$

one - one and onto

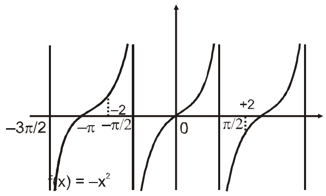
(ii) $f(x) = \frac{x^2}{1+x^2}$ even function $\Rightarrow$ many one

$1 > f(x) \geq 0$ into function

(iii) $f(x) = x^3 - 6x^2 + 11x - 6$

$f(x) = (x - 1)(x - 2)(x - 3)$

D-9.



f(x) = tan (2 sin x)

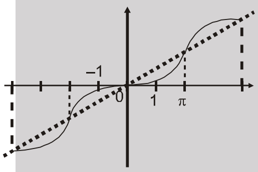
2 sin x $\in$ [–2, 2]

in this interval tan (2 sin x) $\in$ R

$\therefore$ onto function

D-10. f : [–1, 1] $\rightarrow$ [–1, 1]

(i)



f(x) = x – sin x (odd function)

f′(x) = 1 – cos x $\geq$ 0 increasing function

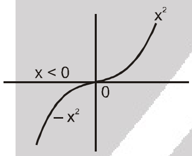
Hence one - one

f(–1) = – 1 + sin 1

f(1) = 1 – sin 1

Range $\equiv$ [–1 + sin 1, 1 – sin 1] $\neq$ co domain function is not bijecive

(ii)



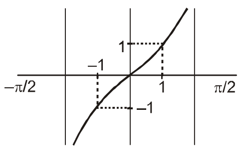
$$f(x) = x|x| = \begin{cases} x^2 & , \quad x \geq 0 \\ -x^2 & , \quad x < 0 \end{cases}$$

one - one function

Range $\equiv$ [–1, 1] = codomain

$\therefore$ onto function

(iii)



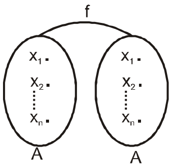
$$f(x) = \tan\left(\frac{\pi x}{4}\right)$$

by graph one-one onto

Bijective function

(iv) f(x) = x^4 even function many-one $\Rightarrow$ Not bijective

D-11.



Total number of functions = n^n;
Total number of one-one onto functions = n!

Section (E)

E-1. **(i)** $f(x) = \sqrt{x^2}$ and $g(x) = (\sqrt{x})^2$ Domain $x \in R$, Domain $x \in [0, \infty)$ non-identical functions

(ii) Domain of f(x) and g(x) are different.

(iii) $f(x) = \sqrt{\dfrac{1+\cos x}{2}}$ and $g(x) = \cos x$

$f(x) = |\cos x|$ non-identical function

(iv) $f(x) = x$ and $g(x) = e^{\ell nx}$, Domain $x \in R^+$

Domain $x \in R$ non-identical function

E-2. $f(x) = \log(x-1) - \log(x-2) = \log\left(\dfrac{x-1}{x-2}\right)$ $x > 1$ and $x > 2$ $\Rightarrow x \in (2, \infty) \Rightarrow$ $g(x) = \log\left(\dfrac{x-1}{x-2}\right)$

$\Rightarrow \dfrac{x-1}{x-2} > 0 \Rightarrow x \in (-\infty, 1) \cup (2, \infty)$ common domain $x \in (2, \infty)$

E- 3. $f(x) = x^2 + x + 1 \Rightarrow$ $g(x) = \sin x \Rightarrow$ $fog(x) = \sin^2 x + \sin x + 1 \Rightarrow gof(x) = \sin(x^2 + x + 1)$
$\therefore$ $fog(x) \neq gof(x)$

E-4. $f(x) = x^2$; $g(x) = \sin x$; $h(x) = \sqrt{x}$
$\Rightarrow$ $fo(goh)(x) = f(\sin\sqrt{x}) = (\sin\sqrt{x})^2 = \sin^2\sqrt{x}$ $(fog)oh(x) = \sin^2(h(x)) = \sin^2\sqrt{x}$

E-5. **(i)** $f(x) = e^x$ and $g(x) = \ell n\, x \Rightarrow$ $fog(x) = e^{\ell n x} = x, x > 0$ $\Rightarrow gof(x) = \ell n\, e^x = x, x \in R$
(ii) $f(x) = |x|$ and $g(x) = \sin x \Rightarrow$ $fog(x) = f(\sin x) = |\sin x| \Rightarrow gof(x) = g(|x|) = \sin|x|$
(iii) $f(x) = \sin x$ and $g(x) = x^2 \Rightarrow$ $fog(x) = \sin(g(x)) = \sin x^2 \Rightarrow gof(x) = (f^2(x)) = (\sin x)^2$
(iv) $f(x) = x^2 + 2$, $g(x) = \dfrac{x}{x-1} \Rightarrow$ $fog(x) = g^2(x) + 2 = \dfrac{x^2}{(x-1)^2} + 2 = \dfrac{3x^2-4x+2}{(x-1)^2}$

$gof(x) = \dfrac{f(x)}{f(x)-1} = \dfrac{x^2+2}{x^2+1}$

E-6. $f(x) = \ell n(x^2 - x + 2)$; $R^+ \to R$ $\Rightarrow$ $g(x) = \{x\} + 1$; $[1, 2] \to [1, 2]$
$f(g(x)) = \ell n(\{x\}^2 + \{x\} + 2)$
Domain [1, 2] for $x \in (1, 2)$ $\{x\} = x - 1$ $\Rightarrow$ $fog(x) = \ell n(x^2 - x + 2)$
Range $[\ell n\, 2, \ell n\, 4)$

E-7. $f(x) = \begin{cases} 1+x^2, & x \le 1 \\ 1+x, & 1 < x \le 2 \end{cases}$ $g(x) = 1 - x, -2 \le x \le 1$

$fog(x) = \begin{cases} 1+g^2, & g(x) \le 1 \Rightarrow x \in [0,1] \\ 1+g(x), & 1 < g(x) \le 2 \Rightarrow x \in [-1,0) \end{cases} \Rightarrow fog(x) = \begin{cases} 1+(1-x)^2, & x \in [0,1] \\ 1+(1-x), & x \in [-1,0) \end{cases}$

$fog(x) = \begin{cases} 2-2x+x^2, & x \in [0,1] \\ 2-x, & x \in [-1, 0) \end{cases}$

E-8. **(i)** $f(g(x)) = \frac{g(x)+2}{g(x)+1}$; $x \neq 0$ & $\frac{x-2}{x} b \neq 1 \Rightarrow x \neq 1$

(ii) $g(f(x)) = \frac{f(x)-2}{f(x)}$; $x \neq -1$ & $f(x) \neq 0 \Rightarrow x \neq -2$

(iii) $f(f(x)) = \frac{f(x)+2}{f(x)+1}$; $x \neq -1$ & $\frac{x+2}{x+1} \neq -1 \Rightarrow x \neq \frac{-3}{2}$

(iv) $f(g(f(x))) = \frac{g(f(x))+2}{g(f(x))+1} \Rightarrow \frac{\frac{f(x)-2}{f(x)}-2}{\frac{f(x)-2}{f(x)}+1}$

$\Rightarrow$ $f(x) \neq 0 \Rightarrow x \neq -2$ also $x \neq -1$

E-9. fof(x) = $f(f(x)) = \begin{cases} \sqrt{2}\, f(x) & f(x) \in Q - \{0\} \\ 3\; f(x) & f(x) \in Q^c \end{cases} = \begin{cases} 3\sqrt{2}\, x & x \in Q - \{0\} \\ 3^2\, x & x \in Q^c \end{cases}$

Similarly fofof(x) = $f(f(f(x))) = \begin{cases} 3\sqrt{2}x\, f(x) & f(x) \in Q - \{0\} \\ 3^2 f(x) & f(x) \in Q^c \end{cases} = \begin{cases} 3^2\sqrt{2}\, x & x \in Q - \{0\} \\ 3^3 x & x \in Q^c \end{cases}$

So $\underbrace{\text{fofo ....f(x)}}_{\text{n times}}$ $\begin{cases} 3^{n-1}\sqrt{2}x & x \in Q - \{0\} \\ 3^n x & x \in Q^c \end{cases}$

E-10. $f(g(x)) = \begin{cases} g(x)+1 & g(x) \leq 4 \\ 2g(x)+1 & 4 < g(x) \leq 9 \\ -g(x)+7 & g(x) > 9 \end{cases}$

$= \begin{cases} x^2+1 & (x^2 \leq 4) \cap (-1 \leq x < 3) \\ (x+2)+1 & (x+2 \leq 4) \cap (3 \leq x < 5) \\ 2x^2+1 & (4 < x^2 \leq 9) \cap (-1 \leq x < 3) \\ 2(x+2)+1 & (4 < x+2 \leq 9) \cap (3 \leq x < 5) \\ -x^2+7 & (x^2 > 9) \cap (-1 \leq x < 3) \\ -(x+2)+7 & (x+2 > 9) \cap (3 \leq x < 5) \end{cases}$

$\Rightarrow$ $f(g(x)) = \begin{cases} x^2+1 & x \in [-1, 2] \\ 2x^2+1 & x \in (2, 3) \\ 2x+5 & x \in [3, 5] \end{cases}$

E-11. $f(x) = \frac{4^x}{4^x+2} \Rightarrow f(1-x) = \frac{4^{1-x}}{4^{1-x}+2} = \frac{2}{2+4^x} \therefore$ f(x) + f(1 – x) = 1

Section (F)

F-1. **(i)** $f(x) = \sin(x^2+1) \Rightarrow$ $f(-x) = f(x)$ = even function

(ii) $f(x) = x + x^2 \Rightarrow$ $f(-x) = x^2 - x \neq f(x)$ or $-f(x)$ Neither even nor odd function

(iii) $f(x) = x\left(\frac{a^x-1}{a^x+1}\right) \Rightarrow$ $f(-x) = -x\left(\frac{a^x-1}{a^{-x}+1}\right) \Rightarrow f(-x) = x\left(\frac{a^x-1}{a^x+1}\right) = f(x)$ even function

(iv) $f(x) = \sin x + \cos x \Rightarrow$ $f(-x) = -\sin x + \cos x \neq f(x)$ or $-f(x)$
Neither even nor odd.

(v) $f(x) = (x^2 - 1)|x| \Rightarrow$ $f(-x) = f(x)$ even function.

(vi) $f(x) = \begin{cases} |\ell n e^x| & ; \quad x \le -1 \\ [2+x]+[2-x] & ; \quad -1 < x < 1 \\ e^{\ell n x} & ; \quad x \ge 1 \end{cases} \Rightarrow f(x) = \begin{cases} |x| & x \le -1 \\ 3 & -1 < x < 0 \\ 4 & x = 0 \\ 3 & 0 < x < 1 \\ x & x \ge 1 \end{cases}$

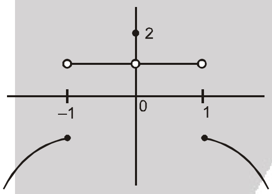
$f(x) = \begin{cases} -x & x \le -1 \\ 3 & -1 < x < 0 \\ 4 & x = 0 \\ 3 & 0 < x < 1 \\ x & x \ge 1 \end{cases}$ b $f(x) = f(-x)$ even function

F-2. **(i)** $f(-x) = \dfrac{(2^x+1)^7}{(2^x)^6}$ neither even nor odd

(ii) $f(-x) = \dfrac{\sec x + x^2 - 9}{x \sin x} = f(x)$ even

(iii) $f(-x) = -f(x)$ odd

(iv) $f(x) = \begin{cases} -x^2 & x \le -1 \\ 2+[x]+[-x] & -1 < x < 1 \\ -x^2 & x \ge 1 \end{cases}$, even by graph of function



F-3. **(i)** Let $f(x) = \sin\sqrt{x}$ is a periodic function with period T (a positive constant)

$\therefore$ $f(x+T) = f(x) \quad \Rightarrow \quad \sin\sqrt{x+T} = \sin\sqrt{x} \quad \Rightarrow \sqrt{x+T} = n\pi + (-1)^n\sqrt{x}$, $n \in I$

since for no value of n, T is indepedent of x which contradicts that $\sin\sqrt{x}$ is a periodic function. Hence it is a non periodic function.

(ii) Let $f(x) = x + \sin x$ is a periodic function with period T.

$x + T + \sin(x+T) = x + \sin x \Rightarrow T + \sin(x+T) = \sin x \Rightarrow T + 2\cos\left(\dfrac{2x+T}{2}\right)\sin\left(\dfrac{T}{2}\right) = 0$

There is no positive constant value of T for which this equation holds true so f(x) is non - periodic function.

F-4. **(i)** $f(x) = 2 + 3\cos(x-2)$ fundamental period $= 2\pi$

(ii) $f(x) = \sin 3x + \cos^2 x + |\tan x| \Rightarrow$ period $\rightarrow \dfrac{2\pi}{3}$, π, π

period of f(x) = L.C.M. $\left(\dfrac{2\pi}{3}, \pi, \pi\right) = 2\pi \Rightarrow$ for fundamental period

$f(x+\pi) = -\sin x + \cos^2 x + |\tan x| \ne f(x)$

$\therefore$ Fundament period $= 2\pi$

(iii) $f(x) = \sin\dfrac{\pi x}{4} + \sin\dfrac{\pi x}{3} \quad \Rightarrow \quad$ period $\rightarrow 8, 6 \quad \Rightarrow \quad$ period of f(x) = L.C.M. (8, 6) = 24

Fundamental period = 24

(iv) $f(x) = \cos\dfrac{3x}{5} - \dfrac{\sin 2x}{7} \Rightarrow$ period $\rightarrow \dfrac{10\pi}{3}$, $7\pi \Rightarrow$ period of f(x) = L.C.M. $\left(\dfrac{10\pi}{3}, 7\pi\right) = 70\pi$

Fundament period $= 70\pi$.

(v) $f(x)=\dfrac{1}{1-\cos x}$ fundamental period = 2π

(vi) $f(x) = \dfrac{\sin 12x}{1+\cos^2 6x}$ $\Rightarrow$ period of f(x) = L.C.M. $\left(\dfrac{\pi}{6},\dfrac{\pi}{3}\right) = \dfrac{\pi}{3}$ for fundamental period

$f\left(x+\dfrac{\pi}{6}\right) = \dfrac{\sin 12\left(x+\dfrac{\pi}{6}\right)}{1+\cos^2 6\left(x+\dfrac{\pi}{6}\right)} = f(x)$ $\therefore$ Fundament period = $\dfrac{\pi}{6}$

(vii) $f(x) = \sec^3 x + \text{cosec}^3 x \Rightarrow$ period $\to 2\pi \Rightarrow 2\pi \Rightarrow$ Fundamental period = L.C.M. $(2\pi, 2\pi)$

Section (G)

G-1. (i) $f : D \to R$

$f(x) = 1 - 2^{-x} \Rightarrow f'(x) = 2^{-x}\,\ell n2 > 0$ increasing function $\Rightarrow$ one one function

D : $[x \in R)$, Range : $(-\infty, 1) \neq$ codomain

$\therefore$ function is not bijective

$\therefore$ f^{-1} does not exist

(ii) $f(x) = (4-(x-7)^3)^{1/5}$

$f'(x) = \dfrac{1}{5}(4-(x-7)^3)^{-4/5}.(-3(x-7)^2) \leq 0$ decreasing function $\Rightarrow$ one one function

$\lim\limits_{x\to\infty} f(x) = -\infty \Rightarrow \lim\limits_{x\to-\infty} f(x) = \infty$

D : R Range : R = codomain $\Rightarrow$ onto function is bijective (invertible)

$y = (4-(x-7)^3)^{1/5} \Rightarrow 4 - y^5 = (x-7)^3$

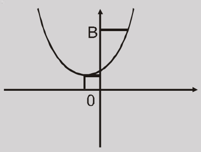
$x = 7 + (4-y^5)^{1/3}$ or $f^{-1}(x) = 7 + (4-x^5)^{1/3}$

(iii) $f(x) = \ell n\left(x+\sqrt{1+x^2}\right)$ D : $x \in R$, Range : R

$y = \ell n\left(x+\sqrt{1+x^2}\right)$ or $x = \dfrac{e^y - e^{-y}}{2} \Rightarrow f^{-1}(x) = \dfrac{e^x - e^{-x}}{2}$

(iv) $f : [0, 3] \to [0, 13]$

$y = f(x) = x^2 + x + 1 \Rightarrow x = \dfrac{-1 \pm \sqrt{1-4(1-y)}}{2} \Rightarrow x = \dfrac{-1\pm\sqrt{4y-3}}{2}$



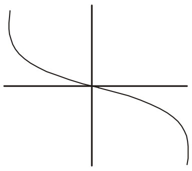
$\therefore$ $f^{-1}(x) = \dfrac{-1+\sqrt{4x-3}}{2}$ as $f^{-1}[1, 13] \to [0, 3]$

G-2. $f(x) = \dfrac{e^{2x}-e^{-2x}}{2}$: R, : R

$2y = e^{2x} - e^{-2x}$ $x = \dfrac{1}{2}\,\ell n\left(y+\sqrt{y^2+1}\right)$ $f^{-1}(x) = \dfrac{1}{2}\,\ell n\left(x+\sqrt{1+x^2}\right)$

G-3. (a) $f(x) = \begin{cases} x^2 & x \leq 0 \\ -x^2 & x > 0 \end{cases}$

clearly $f^{-1}(x) = \begin{cases} \sqrt{-x} & x \leq 0 \\ -\sqrt{x} & x > 0 \end{cases}$ number of solutions of $f(x) = f^{-1}(x)$ is 3.they are –1, 0, 1



(b) $2x^2 - 5x + 2 = \dfrac{5-\sqrt{9+8x}}{4}$ where $x < \dfrac{5}{4}$

$y = 2x^2 - 5x + 2$ and $y = \dfrac{5-\sqrt{9+8x}}{4}$ are inverse function of each other and are not identical hence intersects each other at y = x line only. For intersection point $2x^2 - 5x + 2 = x$

$\Rightarrow \quad x^2 - 3x + 1 = 0 \quad \Rightarrow \quad x = \dfrac{3 \pm \sqrt{5}}{2} \quad$ but $\quad x < \dfrac{5}{4}$

$\therefore \quad x = \dfrac{3-\sqrt{5}}{2}$

G-4. $f(g(x)) = x$; $f'(g(x)).g'(x) = 1 \Rightarrow \quad g'(x) = \dfrac{1}{f'(g(x))}$

$x = g(x)^3 + g(x) + \cos g(x)$

$x = 1 \Rightarrow 1 = g(1)^3 + g(1) + \cos g(1) \Rightarrow g(1) = 0$

so $g'(1) = \dfrac{1}{f'(0)} = 1$

$f'(x) = 3x^2 + 1 - \sin x \Rightarrow \quad f'(0) = 1$

G-5. Clearly $g(g(x)) = \begin{cases} g(x) & g(x) \in Q^c \\ 1-g(x) & g(x) \in Q \end{cases} = \begin{cases} x & x \in Q^c \\ 1-(1-x) & x \in Q \end{cases} = x$ hence

$f(x) = g(x)$ so $\alpha - 1 = 1 \quad \Rightarrow \quad \alpha = 2 \quad$ & when $\alpha = 2$

$-\alpha^2 x + \alpha + 3x - 1 = -4x + 2 + 3x - 1$. hence $\alpha = 2$

Section (H)

H-1. (i) $f(x) = \dfrac{\sin^{-1}x}{x}$. For $\sin^{-1}x$, $x \in [-1, 1]$ and division by zero is undefined $x \neq 0$

$\therefore \quad$ Domain $x \in [-1, 0) \cup (0, 1]$

(ii) $f(x) = \sqrt{1-2x} + 3\sin^{-1}\left(\dfrac{3x-1}{2}\right) \quad 1 - 2x \geq 0$ and $-1 \leq \dfrac{3x-1}{2} \leq 1 \Rightarrow x \leq \dfrac{1}{2}$ and $-\dfrac{1}{3} \leq x \leq 1$

<u>Taking intersection</u>

$\therefore \quad$ Domain $x \in \left[-\dfrac{1}{3}, \dfrac{1}{2}\right]$

(iii) $f(x) = 2^{\sin^{-1}x} - \dfrac{1}{\sqrt{x-2}} \quad -1 \leq x \leq 1$ and $x > 2 \Rightarrow \quad x \in \phi$

H-2. (i) $f(x) = \ell n(\sin^{-1}x) \qquad$ Domain $\sin^{-1}x > 0 \quad \Rightarrow \quad x \in (0, 1]$

Range $0 < \sin^{-1}x \leq \dfrac{\pi}{2} \quad \Rightarrow \quad -\infty < \ell n(\sin^{-1}x) \leq \ell n\left(\dfrac{\pi}{2}\right)$

Inequality doesn't change as ℓn is increasing function

(ii) $f(x) = \sin^{-1}\left(\dfrac{\sqrt{3x^2+1}}{5x^2+1}\right) \quad \because \quad$ it is obvious $\dfrac{\sqrt{3x^2+1}}{5x^2+1}$ is + ve $\forall x \in R$

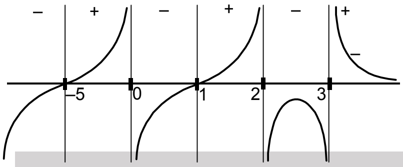
for function to be defined $\dfrac{\sqrt{3x^2+1}}{5x^2+1} \leq 1 \quad \Rightarrow \quad \sqrt{3x^2+1} \leq 5x^2 + 1 \quad \because 5x^2 + 1 > 0 \quad \forall x \in R$

squaring both side $3x^2 + 1 \leq 25x^4 + 10x^2 + 1 \Rightarrow \; 25x^4 + 7x^2 \geq 0$

hold for all $\forall$ x $\in$ R. So $\frac{\sqrt{3x^2+1}}{5x^2+1} \le 1\ \forall$ x $\in$ R at $x \to \infty$ $\frac{\sqrt{3x^2+1}}{5x^2+1} \to 0$

So $0 < \frac{\sqrt{3x^2+1}}{5x^2+1} \le 1 \Rightarrow 0 < \sin^{-1}\frac{\sqrt{3x^2+1}}{5x^2+1} \le \pi/2$

(iii) $f(x) = \cos^{-1}\left(\frac{(x-1)(x+5)}{x(x-2)(x-3)}\right)$



form graph $\frac{(x-1)(x+5)}{x(x-2)(x-3)}$, it is clearly visible

that function $\frac{(x-1)(x+5)}{x(x-2)(x-3)}$ attain all values b/w [–1, 1]

So Range of $\cos^{-1}\left(\frac{(x-1)(x+5)}{x(x-2)(x-3)}\right) \in [0, \pi]$

H-3. **(i)** $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right] = \sin\left[\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)\right] = \sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right) = \sin\left(\frac{3\pi}{6}\right) = 1$

(ii) $\tan\left[\cos^{-1}\frac{1}{2}+\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right] = \tan\left(\frac{\pi}{3}-\tan^{-1}\frac{1}{\sqrt{3}}\right) = \tan\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

(iii) $\sin^{-1}\left[\cos\left\{\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right\}\right] = \sin^{-1}\left[\cos\frac{\pi}{3}\right] = \sin^{-1}\sin\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = \frac{\pi}{6}$

H-4. **(i)** $\sum_{i=1}^{n}\cos^{-1}\alpha_i = 0 \Rightarrow \cos^{-1}\alpha_i = 0 \Rightarrow \alpha_i = 1$

$\sum_{i=1}^{n} i\cdot\alpha_i = 1 + 2 + 3 + \ldots\ldots. + n = n\left(\frac{n+1}{2}\right)$

(ii) $\sum_{i=1}^{2n}\sin^{-1}x_i = n\pi$ we know that $-\frac{\pi}{2} \le \sin^{-1}x_i \le \frac{\pi}{2}$

equility holds good only when $\sin^{-1}x_i = \frac{\pi}{2}\ \forall$ i = 1, 2, 3......2n

$\Rightarrow x_i = 1 \quad \forall \quad i = 1, 2, 3,...., 2n \quad \Rightarrow \quad \sum_{i=1}^{2n} x_i = 2n$

H-5. **(i)** $\cos^{-1}x > \cos^{-1}x^2\ x \in [-1, 1]$.

Defined for $x \in [-1, 1]$

As function decreases

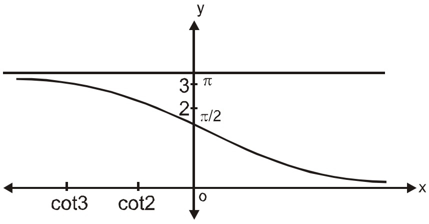
$x < x^2 \Rightarrow x^2 - x > 0$

$x(x-1) > 0 \Rightarrow x \in (-\infty, 0) \cup (1, \infty)$

$x \in [-1, 0)$

(ii) Let arccotx = p $\Rightarrow p^2 - 5p + 6 > 0 \Rightarrow (p-2)(P-3) > 0$

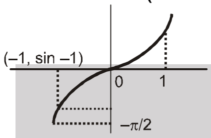
$\cot^{-1}x$ is a decreasing function



$(-\infty, \cot 3) \cup (\cot 2, \infty)$

(iii) $\sin^{-1} x > -1$

$\Rightarrow \quad x \in (-\sin 1, 1]$



(iv) $\cos^{-1} x < 2 \Rightarrow x \in (\cos 2, 1]$

(v) $\cot^{-1} x < -\sqrt{3}$

$\because \quad \cot^{-1} x > 0 \ \forall \ x \in R \Rightarrow$ no solution

H–6. $f : \left[-\frac{\pi}{3}, \frac{\pi}{6}\right] \to B$ if f^{-1} exists, then function should be one-one and onto

$f(x) = 2\cos^2 x + \sqrt{3}\sin 2x + 1 = 2 + \cos 2x + \sqrt{3}\sin 2x = 2 + 2\left(\sin\left(2x + \frac{\pi}{6}\right)\right)$

$\because \quad x \in \left[-\frac{\pi}{3}, \frac{\pi}{6}\right]$

$\therefore \quad 2x + \frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $f(x) \in [0, 4] \Rightarrow f^{-1}(x) = \frac{1}{2}\left(\sin^{-1}\left(\frac{x-2}{2}\right) - \frac{\pi}{6}\right)$

Section (I)

I-1. **(i)** $\sin^{-1}\sin\left(\frac{7\pi}{6}\right) = \sin^{-1}\sin(\pi + \pi/6) = \sin^{-1}\sin(-\pi/6)$

(ii) $\tan^{-1}\tan\frac{2\pi}{3} = \tan^{-1}\tan\left(\pi - \frac{\pi}{3}\right) = -\frac{\pi}{3}$

(iii) $\cos^{-1}\cos\left(\pi + \frac{\pi}{4}\right) = \cos^{-1}\left(-\cos\frac{\pi}{4}\right) = \pi - \cos^{-1}\cos\frac{\pi}{4} = \pi - \frac{\pi}{4} = \frac{3\pi}{4} = -\pi/6$

(iv) $\sec^{-1}\sec(2\pi - \pi/4) = \sec^{-1}\sec\pi/4 = \pi/4$

I-2. **(i)** $\sin^{-1}(\sin 4) = (\pi - 4) \qquad [\because \pi < 4 < \frac{3\pi}{2}]$

(ii) $\cos^{-1}(\cos 10) = 4\pi - 10$

(iii) $\tan^{-1}(\tan(-6)) = \tan^{-1}(-(\tan(6))) = -\tan\tan^{-1} 6 = -(6 - 2\pi) = 2\pi - 6$

(iv) $\cot^{-1}(\cot(-10)) = \pi - \cot^{-1}\cot 10 = \pi - (10 - 3\pi) = 4\pi - 10$

(v) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\left(\cos\frac{9\pi}{10} - \sin\frac{9\pi}{10}\right)\right) = \cos^{-1}\cos\left(\frac{\pi}{4} + \frac{9\pi}{10}\right) = \cos^{-1}\cos\frac{23\pi}{20} = \cos^{-1}\left[-\cos\frac{3\pi}{20}\right] = \pi\frac{3\pi}{20} = \frac{17\pi}{20}$

I-3. **(i)** $\cot(\tan^{-1} a + \cot^{-1} a) = \cot\frac{\pi}{2} = 0$

(ii) $\sin(\sin^{-1}x + \cos^{-1}x) = \sin\frac{\pi}{2} = 1$

I-4. $\tan^{-1}x > \frac{\pi}{2} - \tan^{-1}x \Rightarrow \tan^{-1}x > \frac{\pi}{4} \Rightarrow x > \tan\frac{\pi}{4} \Rightarrow x > 1$

Section (J)

J-1. **(i)** $\sin\cos^{-1}\frac{3}{5} = \sin\left(\sin^{-1}\frac{4}{5}\right) = \frac{4}{5}$ **(ii)** $\tan\left(\cos^{-1}\frac{1}{3}\right) = \tan(\tan^{-1}2\sqrt{2}) = 2\sqrt{2}$

(iii) $\operatorname{cosec}\left(\sec^{-1}\frac{\sqrt{41}}{5}\right) = \text{cossec}\ \operatorname{cosec}^{-1}\frac{\sqrt{41}}{4} = \frac{\sqrt{41}}{4}$

(iv) $\tan\left(\operatorname{cosec}^{-1}\frac{65}{63}\right) = \tan\tan^{-1}\frac{63}{16} = \frac{63}{16}$

(v) $\sin = \left(\frac{\pi}{6} + \cos^{-1}\frac{1}{4}\right) = \sin\frac{\pi}{6}\ \cos\cos^{-1}\frac{1}{4} + \cos\frac{\pi}{6}\ \sin\cos^{-1}\frac{1}{4} = \frac{1}{2}\times\frac{1}{4} + \frac{\sqrt{3}}{2}\times\frac{\sqrt{15}}{4} = \frac{1+3\sqrt{5}}{8}$

(vi) $\cos\left(\sin^{-1}\frac{4}{5} + \cos^{-1}\frac{2}{3}\right) = \cos\sin^{-1}\frac{4}{5}\ \cos\cos^{-1}\frac{2}{3} - \sin\sin^{-1}\frac{4}{3}\ \sin\cos^{-1}\frac{2}{3} = \frac{3}{5}\times\frac{2}{3} - \frac{4}{5}\times\frac{\sqrt{5}}{3} = \frac{6-4\sqrt{5}}{15}$

(vii) $\sec\left[\tan\left\{\tan^{-1}\left(-\frac{\pi}{3}\right)\right\}\right] = \sec\tan\tan^{-1}\frac{\pi}{3} = \sec\frac{\pi}{3} = 2$

(viii) $\cos\tan^{-1}\frac{2}{\sqrt{5}} = \cos\cos^{-1}\frac{\sqrt{5}}{3} = \frac{\sqrt{5}}{3}$

(ix) $\tan\left(\frac{\pi}{2} - \sec^{-1}3\right) = \cot\sec^{-1}3 = \cot\cot^{-1}\frac{1}{\sqrt{8}} = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$

J-2. $\sin^{-1}(\cos(\sin^{-1}x)) + \cos^{-1}\sin(\cos^{-1}x) \Rightarrow \sin^{-1}\left(\sqrt{1-x^2}\right) + \cos^{-1}\sqrt{1-x^2} = \frac{\pi}{2}$

J-3. $\tan^{-1}x + \tan^{-1}y = \pi - 2\tan^{-1}z$

$\frac{x+y}{1-xy} = -\tan(2\tan^{-1}z) \Rightarrow \frac{x+y}{1-xy} = -\frac{2z}{1-z^2}$

$\Rightarrow x + y - xz^2 - yz^2 = -2z + 2xyz \Rightarrow x + y + 2z = xz^2 + yz^2 + 2xyz$

J-4. $\frac{\pi}{2} + \sin^{-1}x + \frac{3\pi}{2} + \tan^{-1}y = 4\sec^{-1}z + 5\operatorname{cosec}^{-1}z \Rightarrow \sin^{-1}x + \tan^{-1}y = \operatorname{cosec}^{-1}z$

$\Rightarrow \tan^{-1}\frac{x}{\sqrt{1-x^2}} + \tan^{-1}y = \tan^{-1}\frac{1/z}{\sqrt{1-\frac{1}{z^2}}} \Rightarrow \frac{\frac{x}{\sqrt{1-x^2}} + y}{1 - \frac{xy}{\sqrt{1-x^2}}} = \frac{1}{\sqrt{z^2-1}} \Rightarrow \sqrt{z^2-1} = \frac{\sqrt{1+x^2} - xy}{x + y\sqrt{1-x^2}}$

J-5. $g(x) = \begin{cases} \tan^{-1}\frac{2}{x}, & x > 0 \\ \frac{\pi}{2}, & x = 0 \\ -\tan^{-1}\frac{2}{x}, & x < 0 \end{cases}$ $\quad x > 0\ h(f(x)) = \frac{x}{2} = h(g(x))$

$x < 0\ h(f(x)) = \frac{x}{2}$, $h(g(x)) = \frac{-x}{2}$

J-6. **(i)** Let $\tan^{-1}x = \theta \Rightarrow \tan\theta = x \quad \cot\theta = \frac{1}{x}\ \forall\ x > 0$

$\theta = -\pi + \cot^{-1}\frac{1}{x}\ \forall \quad x < 0$

$\sin\theta = \frac{x}{\sqrt{1+x^2}} \Rightarrow \theta = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$\cos\theta = \dfrac{1}{\sqrt{1+x^2}}$ $\quad x > 0$

$\theta = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}} = \tan^{-1} x$ $\quad x > 0$

and for $x < 0 \Rightarrow \cos^{-1}\cos\theta = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}} \Rightarrow -\theta = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}}$

$\Rightarrow \tan^{-1}x = -\cos^{-1}\dfrac{1}{\sqrt{1+x^2}} \Rightarrow \tan^{-1} x = -\pi + \cot^{-1}\dfrac{1}{x} = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}} = -\cos^{-1}\dfrac{x}{\sqrt{1+x^2}}$ where $x < 0$

(ii) Let

$\theta = \cos^{-1} x$ given $-1 < x < 0$ $\Rightarrow$ $\cos\theta = x$ $\quad \theta \in (\frac{\pi}{2}, \pi)$

$\sec\theta = \dfrac{1}{x}$ $\quad \theta = \sec^{-1}\dfrac{1}{x}$ $\Rightarrow$ $\sin\theta = \sqrt{1-x^2} \Rightarrow \theta = \pi - \sin^{-1}\sqrt{1-x^2}$

$\tan\theta = \dfrac{\sqrt{1-x^2}}{x}$ $\Rightarrow$ $\theta = \pi + \tan^{-1}\dfrac{\sqrt{1-x^2}}{x}$

$\cot\theta = \dfrac{x}{\sqrt{1-x^2}}$ $\Rightarrow$ $\theta = \cot^{-1}\dfrac{x}{\sqrt{1-x^2}}$

J-7. **(i)** Let $\tan^{-1} x = \theta \Rightarrow$ $\tan^{-1}\dfrac{2x}{1-x^2} = \tan^{-1}\tan 2\theta = 2\theta - \pi = 2\tan^{-1}x - \pi$

(ii) Let $\sin^{-1} x = \theta \Rightarrow \sin^{-1}(2x\sqrt{1-x^2}) = \sin^{-1}\sin 2\theta = \pi - 2\theta = \pi - 2\sin^{-1}x$

(iii) Let $\cos^{-1} x = \theta \Rightarrow \cos^{-1}(2x^2 - 1) = \cos^{-1}\cos 2\theta = 2\pi - 2\theta = 2\pi - 2\cos^{-1}x$

J-8. Let $x = \tan\theta$ and $y = \tan\alpha$ $\Rightarrow$ $\sin^{-1}\dfrac{2x}{1+x^2} = \sin^{-1}\sin 2\theta = \pi - 2\theta$

$\cos^{-1}\left(\dfrac{1-y^2}{1+y^2}\right) = \cos^{-1}\cos 2\alpha = 2\alpha$ $\Rightarrow$ $\tan\left(\dfrac{\pi}{2} - \theta + \alpha\right) = \cot(\theta - \alpha) = \cot[\tan^{-1} x - \tan^{-1}y]$

$\cot\left(\tan^{-1}\dfrac{x-y}{1+xy}\right) = \cot\left(\cot^{-1}\left(\dfrac{1+xy}{x-y}\right)\right) = \dfrac{1+xy}{x-y}$

J-9. **(i)** Let $\sin^{-1} x = \theta$

$\cos(2\theta) = \dfrac{1}{3}$ $\Rightarrow$ $1 - 2\sin^2\theta = \dfrac{1}{3}$ $\sin^2\theta = \dfrac{1}{3} \Rightarrow x = \sin\theta = \pm\dfrac{1}{\sqrt{3}}$

(ii) $\cot^{-1} x + \tan^{-1} x = \dfrac{\pi}{2}$. so $x = 3$

(iii) $\tan^{-1}\left(\dfrac{x-1}{x-2}\right) + \tan^{-1}\left(\dfrac{x+1}{x+2}\right) = \tan^{-1}\left[\dfrac{\dfrac{x-1}{x-2} + \dfrac{x+1}{x+2}}{1 - \dfrac{x^2-1}{x^2-4}}\right] = \tan^{-1}\left(\dfrac{4-2x^2}{3}\right) = \dfrac{\pi}{4}$

case- I

$\dfrac{4-2x^2}{3} = 1$ $\Rightarrow$ $x = \pm\dfrac{1}{\sqrt{2}}$ ... (1)

If $\dfrac{(x-1)(x+1)}{(x-2)(x+2)} < 1 \Rightarrow$ $x \in (-2, 2)$ ...(2)

from (1) & (2) $x = \pm\dfrac{1}{\sqrt{2}}$

case-II If $\frac{(x-1)(x+1)}{(x-2)(x+2)} > 1$

$\Rightarrow \quad x \in (-\infty, -2) \cup (2, \infty)$ ....(3)

$\tan^{-1}\left(\frac{4-2x^2}{3}\right) + \pi = \frac{\pi}{4}$

$\frac{4-2x^2}{3} = 1 \qquad x = \pm\frac{1}{\sqrt{2}}$ .....(4)

from (3) & (4) $\Rightarrow \quad x \in \phi$

case-III $\therefore \quad \frac{(x-1)(x+1)}{(x-2)(x+2)} = 1$

no solution $\Rightarrow \quad x = \pm\frac{1}{\sqrt{2}}$ are the solutions

(iv) $\sin^{-1} x = \frac{2\pi}{3} - \sin^{-1} 2x \quad \Rightarrow \quad \sin x \sin^{-1} x = \sin\left(\frac{2\pi}{3} - \sin^{-1} 2x\right)$

$x = \frac{\sqrt{3}}{2}\sqrt{1-4x^2} + \frac{2x}{2} \quad \Rightarrow \quad x = \frac{\sqrt{3}}{2}\sqrt{1-4x^2} + x$

$1 - 4x^2 = 0 \; x = \pm\frac{1}{2}$ But $x = -\frac{1}{2}$ does not satisfy $\Rightarrow \quad x = \frac{1}{2}$

Section (K)

K-1. (i) $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \sin^{-1}\left[\frac{3}{5}\times\frac{15}{17} + \frac{8}{17}\times\frac{4}{5}\right] = \sin^{-1}\frac{77}{85}$

(ii) $\tan^{-1}\frac{3}{4} + \tan\frac{5}{12} = \tan^{-1}\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\times\frac{5}{12}} = \tan^{-1}\frac{56}{33} = \cos^{-1}\frac{33}{65}$

(iii) $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \cot^{-1} 3 = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \sin^{-1}\frac{1}{\sqrt{10}} = \sin^{-1}\left[\frac{3}{\sqrt{5}\sqrt{10}} + \frac{1}{\sqrt{10}}.\frac{2}{\sqrt{5}}\right] = \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

(iv) $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}.\frac{1}{8}}\right) + \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right)$

$= \tan^{-1}\left(\frac{11}{23}\right) + \tan^{-1}\left(\frac{12}{34}\right) = \tan^{-1}\left(\frac{11}{23}\right) + \tan^{-1}\left(\frac{6}{17}\right) = \tan^{-1}\left(\frac{\frac{11}{23}+\frac{6}{17}}{1-\frac{11}{23}\times\frac{6}{17}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

K-2. (i) $\tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7}$ +......... upto n terms

$= \sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+r)(x+r-1)}\right) = \sum_{r=1}^{n} \tan^{-1}\left(\frac{(x+r)-(x+r-1)}{1+(x+r)(x+r-1)}\right) = \sum_{r=1}^{n}\left\{\tan^{-1}(x+r) - \tan^{-1}(x+r-1)\right\}$

$= [\{\tan^{-1}(x+1) - \tan^{-1} x\} + \{\tan^{-1}(x+2) - \tan^{-1}(x+1)\} + \{\tan^{-1}(x+3) - \tan^{-1}(x+2)\} + ...+$
$\{\tan^{-1}(x+n) - \tan^{-1}(x+n-1)\}] = \tan^{-1}(x+n) - \tan^{-1} x$

(ii) $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{2}{9} + ... + \tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}}$ + ... + upto infinite terms

$= \sum_{n=1}^{\infty} \tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}} = \sum_{n=1}^{\infty} \tan^{-1}\frac{2^n - 2^{n-1}}{1+2^n.2^{n-1}} = \sum_{n=1}^{\infty}\{\tan^{-1}2^n - \tan^{-1}2^{n-1}\}$

$= [(\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}2^2 - \tan^{-1}2) +$

$(\tan^{-1} 2^3 - \tan^{-1} 2^2) + ... + \lim_{n\to\infty} (\tan^{-1} 2^n - \tan^{-1} 2^{n-1})] = \lim_{n\to\infty} \tan^{-1} 2^n - \tan^{-1} 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

(iii) $\sin^{-1}\frac{1}{\sqrt{2}} + \sin^{-1}\frac{\sqrt{2}-1}{\sqrt{6}} + ... + \sin^{-1}\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n-1)}} + ...$ up to infinite terms

$$= \sum_{n=1}^{\infty} \sin^{-1}\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}} = \sum_{n=1}^{\infty}\left\{\sin^{-1}\frac{1}{\sqrt{n}} - \sin^{-1}\frac{1}{\sqrt{n+1}}\right\}$$

$$= \left[\left(\sin^{-1}1 - \sin^{-1}\frac{1}{\sqrt{2}}\right) + \left(\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}\frac{1}{\sqrt{3}}\right) + \left(\sin^{-1}\frac{1}{\sqrt{3}} - \sin^{-1}\frac{1}{\sqrt{4}}\right) + ... + \lim_{n\to\infty}\left(\sin^{-1}\frac{1}{\sqrt{n}} - \sin^{-1}\frac{1}{\sqrt{n+1}}\right)\right] = \frac{\pi}{2}$$

PART - II

Section (A)

A-1. B U C = {c, d, e} , $B \cap C$ = {d}
$A \cup (B \cup C) = \phi$
$A \cup (B \cap C)$ = {a, b, d}
$A \times (B \cup C)$ = {(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)}
$A \times (B \cap C)$ = {(a, d), (b, d)}

A-2. n(A) = 3, n(B) = 2; n(C) = 3 $\Rightarrow$ $n(A \times B \times C)$ = 3.2.3. = 18.

A-3. A × B is a relation defined from set A to set B

A-4. Obviously $R \subseteq A \times B$

A-5. $R_1 \rightarrow$ Domain = {1, 3, 5}
Range = {3, 5, 7} so R_1 is a relation
$R_2 \rightarrow$ Domain = {1,2, 3, 4, 5}
Range = {1, 3, 5} so R_2 is a relation

$R_3 \rightarrow$ Domain = {1, 3, 5}
Range = {1, 3, 5, 7} so R_3 is a relation

$R_4 \rightarrow$ Domain = {1, 2, 7} $\not\subset$ X so R_4 is not a relation

A-6. A = {1, 2, 3}
$-5 \le a^2 - b^2 \le 5$
R = {(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)}
Domain of R = {1, 2, 3}
Range of R = {1, 2, 3}

Section (B)

B-1. For any $a \in N$, we find that a is divisible by a, therefore R is reflexive but R is not symmetric, because aRb does not imply that bRa.

B-2. Since $x \nless x$, therefore R is not reflexive. Also x < y does not imply that y < x, So R is not symmetric. Let xRy and yRz. Then x < y and y < z $\Rightarrow$ x < z i.e., xRz. Hence R is transitive.

B-3. $xR_2y \Leftrightarrow x \ge y$ is not symmetric relation
$xR_3y \Leftrightarrow x / y$ is not symmetric relation
$xR_4y \Leftrightarrow x < y$ is not symmetric relation
$xR_1y \Leftrightarrow x^2 = y^2$ is reflexive, symmetric and transitive so equivalence relation
$xR_4y \Leftrightarrow x < y$ is not symmetric relation
$xR_1y \Leftrightarrow x^2 = y^2$ is reflexive, symmetric and transitive so equivalence relation

B-4. For any $a \in R$, we have $a \geq a$, Therefore the relation R_1 is reflexive but it is not symmetric as $(2, 1) \in R_1$ but $(1, 2) \notin R_1$. The relation R_1 is transitive also, because $(a, b) \in R_1$, $(b, c) \in R_1$ imply that $a \geq b$ and $b \geq c$ which is turn imply that $a \geq c \Rightarrow (a, c) \in R_1$.

B-5. Here $\alpha R \beta \Leftrightarrow \alpha \perp \beta \Leftrightarrow \beta \perp \alpha$
Hence R is symmetric.

B-6. $1 + a.a = 1 + a^2 > 0$, $\forall\ a \in S$, $\therefore$ $(a, a) \in R$
$\therefore$ R is reflexive
$(a, b) \in R \Rightarrow 1 + ab > 0 \Rightarrow$ $1 + ba > 0 \Rightarrow$ $(b, a) \in R$
$\therefore$ R is symmetric.
$\because$ $(a, b) \in R$ and $(b, c) \in R$ need not imply $(a, c) \in R$
Hence, R is not transitive.

B-7. **1.** R is not symmetric so it is incorrect.
2. $S_1 \neq S_2$ so not reflexive
Let $S_1 = \{1, 2, 3\}$ & $S_2 = \{1, 2\}$
it satisfies the condition
$S_1 \not\subset S_2 \Rightarrow S_2 \not\subset S_1$
So non symmetric.
let $S_1 = \{1, 2\}$, $S_2 = \{4, 5\}$, $S_3 = \{1, 2, 3\}$
as $S_1 \not\subset S_2$ and $S_1 \not\subset S_3$ S_1 S_3
so non transitive.

B-8. We have $(a, b)R (a, b)$ for all $(a, b) \in N \times N$
Since $a + b = b + a$. Hence, R is reflexive.
R is symmetric for all $(a, b), (c, d) \in N \times N$ we have $(a, b) R (c, d)$
$\Rightarrow$ $a + d = b + c$
$\Rightarrow$ $c + b = d + a$ $\Rightarrow$ $(c, d) R (a, b)$.
$(a, b)R (c, d)$ and $(c, d)R (e, f)$
$a + d = b + c$ and $c + f = d + e$,
$\Rightarrow$ $a + d + c + f = b + c + d + e \Rightarrow a + f = b + e$
$\Rightarrow (a, b) R (e, f) \Rightarrow$ R is transitive
Thus, $(a, b) R (c, d)$ and $(c, d) R (e, f)$ $\Rightarrow$ $(a, b) R (e, f)$

B-9. $\ell_1 \parallel \ell_2 \Rightarrow$ R is reflexive.
$\ell_1 \parallel \ell_2 \Rightarrow \ell_2 \parallel \ell_1$ $\therefore$ R is symmetric.
$\ell_1 \parallel \ell_2 \Rightarrow \ell_2 \parallel \ell_3 \Rightarrow \ell_1 \parallel \ell_3$ $\therefore$ R is transitive.

B-10. $R = \{(x, y) ; x, y \in A, x + y = 5\}$ $A = \{1, 2, 3, 4, 5\}$
$R = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$
R is symmetric but neither reflexive nor transitive.

B-11. Reflexive Relation :
$A.A \neq 0$ for $\forall A \in S$ so Relation is not Reflexive Relation
Symmetric Relation :
$A.B = 0 \Rightarrow BA = 0$ Not True $\forall$ $A, B \in S$
for example $A = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ $\Rightarrow AB = 0$ but $BA \neq 0$
Transitive Relation :
$AB = 0$, $BC = 0 \Rightarrow AC = 0$ Not True, $\forall$ $A, B, C \in S$
for example $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ $\Rightarrow AB = 0$, $BC = 0$ but $AC \neq 0$

Section (C)

C-1. $[e] - [-\pi]$

here $e \approx 2.7$

$\pi \approx 3.14$

$\therefore$ $[2.7] - [-3.14] = 2 - (-4) = 2 + 4 = 6$

C-2. $2\{x\}^2 - 5\{x\} + 2 = 0$

So $\{x\} = \frac{1}{2}, 2$ but $\{x\} \neq 2$

So $\{x\} = \frac{1}{2}$ x has infinite values.

C-3. $\sum_{n=1}^{49} f(n) = 0;$ $\sum_{n=50}^{149} f(n) = 100;$ $f(150) + f(151) = 4;$ $\sum_{n=1}^{151} f(n) = 104$

C-4. Let $2x = t$

then $[t] - 3\{t\} = 1$

Now put $t = I + f$; $0 \le f < 1 \Rightarrow I - 3f = 1$

$0 \le I - 1 < 3$

$1 \le I < 4 \Rightarrow I = 1, 2, 3$

When $I = 1$, then $= 0$ $\therefore$ $x = \frac{1}{2}$

When $I = 2$, then $f = \frac{1}{3}$ $\therefore$ $x = \frac{7}{6}$

When $I = 3$, then $f = \frac{2}{3}$ $\therefore$ $x = \frac{11}{6}$

so it has three solutions.

C-5. $\frac{x^2 - 5x + 4}{\{x\}} < 0 \Rightarrow (x-1)(x-4) < 0$ and $x \notin I$ $\Rightarrow x \in (1, 4) - \{2, 3\}$.

C-6. $x^3 - 4x + 3x > 0 \Rightarrow x \in (0, 1) \cup (3, \infty)$

Since $x \in Z$, x is 4, 5, 6, 7, 8, 9, 10

Hence number of values of x is 7.

C-7. **Case :** $x \in I$

$0 = |x - 1| \Rightarrow x = 1$

Case- : $x \notin I$

$1 = |x-1| \Rightarrow x = 0, 2$ (reject both) $\Rightarrow x = 1$

C-8. p should be irrational (D).

Section (D)

D-1. For domain $-\log_{0.3}(x-1) \ge 0$ and $x^2 + 2x + 8 > 0$

$\Rightarrow$ $\log_{0.3}(x-1) \le 0$ $\Rightarrow$ $(x+1)^2 + 7 > 0$

$\Rightarrow$ $(x-1) \ge 1$ $\Rightarrow$ $x \in R$

$\Rightarrow$ $x \ge 2$ $\therefore$ Taking intersection $x \in [2, \infty)$

D-2. $f(x) = \log_e(3x^2 - 4x + 5)$

$3x^2 - 4x + 5 \ge \frac{11}{3} \Rightarrow \ell n(3x^2 - 4x + 5) \ge \ell n \frac{11}{3}$ [$\because$ ℓn is an increasing function]

$\therefore$ Range is $\left[\ell n \frac{11}{3}, \infty\right)$

D-3. $f(x) = 4^x + 2^x + 1$

Let $2^x = t > 0, \ \forall\ x \in R$

$\therefore \quad f(x) = g(t) = t^2 + t + 1, \quad t > 0$

$g(t) = \left(t+\frac{1}{2}\right)^2 + \frac{3}{4} \quad \Rightarrow \quad \left(t+\frac{1}{2}\right) > \frac{1}{2} \quad \Rightarrow \left(t+\frac{1}{2}\right)^2 > \frac{1}{4} \Rightarrow \quad \left(t+\frac{1}{2}\right)^2 + \frac{3}{4} > 1$

Range is $(1, \infty)$

D-4. $f(x) = \log_{\sqrt{5}}\left(\sqrt{2}(\sin x - \cos x) + 3\right)$ we know that $-\sqrt{2} \le \sin x - \cos x \le \sqrt{2}, \ \forall\ x \in R$ [since $-\sqrt{a^2+b^2} \le a \sin x + b \cos x \le \sqrt{a^2+b^2}$] $\quad \Rightarrow \quad -2 \le \sqrt{2}\ (\sin x - \cos x) \le 2$

$\Rightarrow \quad 1 \le \sqrt{2}\ (\sin x - \cos x) + 3 \le 5 \Rightarrow 0 \le \log_{\sqrt{5}}\ (\sqrt{2}\ (\sin x - \cos x) + 3) \le 2$. Hence range is [0, 2]

D-5. <u>**One One / Many One**</u>

$f(x) = \frac{2x^2 - x + 5}{7x^2 + 2x + 10}$, Domain $x \in R \quad \Rightarrow \quad f'(x) = \frac{(4x-1)(7x^2+2x+10) - (14x+2)(2x^2-x+5)}{(7x^2+2x+10)^2}$

$f'(x) = \frac{11x^2 - 30x - 20}{(7x^2+2x+10)^2} > 0 \quad \Rightarrow \quad x \in (-\infty, 0) \cup \left(\frac{30}{11}, \infty\right)$

$f'(x) < 0 \ \Rightarrow\ x \in \left(0, \frac{30}{11}\right) \quad \Rightarrow \quad f'(x) = 0 \quad \Rightarrow \quad x = 0, \frac{30}{11}$

Function is increasing and decreasing in different intervals, so non monotonic

$\therefore$ Many one function.

<u>**Onto / Into**</u>

$f(x) = \frac{2x^2 - x + 5}{7x^2 + 2x + 10} \quad \Rightarrow 2x^2 - x + 5 > 0, \ \forall\ x \in R$ and $7x^2 + 2x + 10 > 0 \ \ \forall\ x \in R$

$\because\ a = 2 > 0$ and $\quad \because \quad a = 7$ and $D = 4 - 280 < 0$

$D = 1 - 40 = -39 < 0 \quad \therefore \quad f(x) > 0\ \forall\ x \in R$

Also f(x) never tends to $\pm\infty$ as $7x^2 + 2x + 10$ has no real roots, Range $\ne$ Codomain so into function.

D-6. $f(x) = x^3 + x^2 + 3x + \sin x,\ x \in R \Rightarrow f'(x) = 3x^2 + 2x + 3 + \cos x$

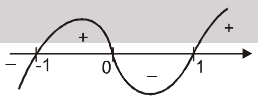
$\because \quad 3x^2 + 2x + 3 \ge \frac{32}{12}$ as $a = 3 > 0$ and $D < 0$

$\Rightarrow \quad -1 \le \cos x \le 1$

so $f'(x) > 0\ \forall\ x \in R \quad \Rightarrow \quad \lim\limits_{x\to\infty} f(x) = +\infty \quad \Rightarrow \quad \lim\limits_{x\to\infty} f(x) = -\infty$

Hence f(x) is one-one and onto function (as f(x) is continuous function)

D-7. $4 - x^2 \ne 0,\ x^3 - x > 0 \quad \Rightarrow \quad x \ne \pm 2$ and $-1 < x < 0$ or $1 < x < \infty$



$\therefore\ \ D = (-1, 0) \cup (1, \infty) - \{2\} \quad$ or $D = (-1, 0) \cup (1, 2) \cup (2, \infty)$.

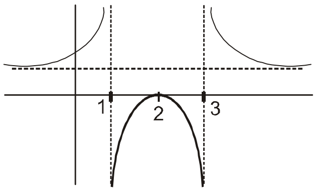
D-8. $f : [0, \infty) \to [0, \infty) \Rightarrow \quad f(x) = \frac{x}{1+x}$

$\frac{x_1}{1+x_1} = \frac{x_2}{1+x_2} \quad \Rightarrow \quad x_1 = x_2$ only

for given domain $f(x) < 1$

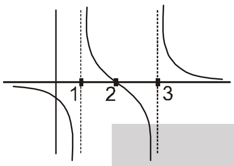
$\therefore \quad$ function is into

D-9.



$$\frac{\left(1-\frac{2}{x}\right)^2}{\left(1-\frac{1}{x}\right)\left(1-\frac{3}{x}\right)} \to 1 \text{ when } x \to \infty$$

D-10.



$$f(x) = \frac{x-2}{(x-1)(x-3)}$$

D-11. $f(g(x_1)) = f(g(x_2)) \Rightarrow g(x_1) = g(x_2)$
as f is one - one function $\Rightarrow x_1 = x_2$
as g is one - one function
hence $f(g(x_1)) = f(g(x_2)) \Rightarrow x_1 = x_2 \Rightarrow$ f(x) is one - one function

D-12. If f(x) is increasing continuous function in $[\alpha, \beta]$, then its range is $[f(\alpha), f(\beta)]$ but for discontinuous function the statement is not true.
So D is correct.

D-13. $y = (f - g)(x) = \begin{cases} -x, & x \in Q \\ x, & x \notin Q \end{cases}$

Which is one-one and onto function

Section (E)

E-1. **(A)** $f(x) = \sin^2x + \cos^2x,\ x \in R$ and $g(x) = 1,\ x \in R$
$f(x) = 1,\ x \in R$ and $g(x) = 1,\ x \in R$ identical functions

(B) $f(x) = \sec^2x - \tan^2x,\ x \in R - (2n+1)\frac{\pi}{2}$ and $g(x) = 1,\ x \in R$ Non-identical functions

(C) $f(x) = \text{cosec}^2x - \cot^2x,\ x \in R - n\pi$ and $g(x) = 1,\ x \in R$ Non-identical functions

E-2. Domain of f(g(x)). Range of $g(x) \equiv$ Domain of f(x)
$\Rightarrow -5 \le |2x+5| \le 7 \Rightarrow 0 \le |2x+5| \le 7 \Rightarrow -7 \le 2x+5 \le 7$
$\Rightarrow -12 \le 2x \le 2 \Rightarrow -6 \le x \le 1$

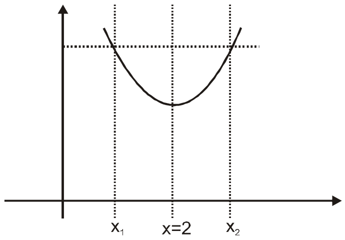
E-3. $g(x) = 1 + \{x\} \Rightarrow f(x) = \text{sgn}(x)$
$\therefore f(g(x)) = \text{sgn}(1 + \{x\}) = 1$

Section (F)

F-1. $f(x) = \log\left(\frac{1+\sin x}{1-\sin x}\right) \Rightarrow f(-x) = \log\left(\frac{1-\sin x}{1+\sin x}\right) = -\log\left(\frac{1+\sin x}{1-\sin x}\right) = -f(x)$ odd function

F-2. $f(x) = [x] + \frac{1}{2},\ x \notin I \Rightarrow f(-x) = [-x] + \frac{1}{2} = -[x] - 1 + \frac{1}{2} = - \;= -\left([x]+\frac{1}{2}\right) = -f(x)$ odd function

F-3. Let us consider a graph symmetric w.r.t. line x = 2 as shown in figure



from figure $f(x_1) = f(x_2)$ where $x_1 = 2 - x$ & $x_2 = 2 + x$ $\quad\therefore\ f(2-x) = f(2+x)$

F-4. $f(x) = \sec(\sin x)$. Since $\sin x$ is a periodic function with fundamental period 2π. $f(x)$ has a period 2π for fundamental period $f(x+\pi) = \sec(\sin(\pi + x)) = \sec(-\sin x) = \sec(\sin x) = f(x)$

$f\left(x+\frac{\pi}{2}\right) \neq f(x)$ hence fundamental period is π

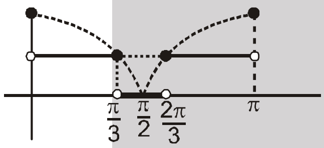
F-5. $f(x) = \sin\left(\sqrt{[a]}\, x\right) \quad\Rightarrow\quad \text{Period} = \frac{2\pi}{\sqrt{[a]}} = \pi$

$[a] = 4 \quad\Rightarrow\quad a \in [4, 5)$

F-6. $y = [2\,|\cos x|]$

period $= \pi$

area in $[0, \pi] = 2.\ \frac{\pi}{3}$



so required area $= 9 \times \frac{2\pi}{3} = 6\pi$

Section (G)

G-1. $\frac{y}{1} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

By compnendo and dividendo

$\frac{1+y}{1-y} = \frac{2e^x}{2e^{-x}} \Rightarrow 2x = \ell n\left(\frac{1+y}{1-y}\right)$

$\therefore \quad f^{-1}(x) = \frac{1}{2}\,\ell n\left(\frac{1+x}{1-x}\right)$

G-2. $f : [1, \infty) \to [2, \infty)$

$y = f(x) = x + \frac{1}{x} \quad\Rightarrow\quad x^2 - xy + 1 = 0 \quad\Rightarrow\quad x = \frac{y \pm \sqrt{y^2 - 4}}{2}$

$\therefore \quad f^{-1}(x) = \frac{x + \sqrt{x^2-4}}{2}$ as $f^{-1} : [2, \infty) \to [1, \infty)$

G-3. Since $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y = -x$.

If (α, β) lies on $y = f(x)$ then $(-\beta, -\alpha)$ on $y = f^{-1}(x)$

$\Rightarrow (-\beta, -\alpha)$ lies on $y = f(x)$

$\Rightarrow y = f(x)$ is odd.

G-4. $f(x) = \frac{ax+b}{cx+d} \Rightarrow fof(x) = \frac{a\left(\frac{ax+b}{cx+d}\right)+b}{c\left(\frac{ax+b}{cx+d}\right)+d} \Rightarrow fof(x) = \frac{a^2x + ab + bcx + bd}{acx + bc + cdx + d^2}$

$\text{fof}(x) = \dfrac{(a^2+bc)x+(ab+bd)}{(ac+cd)x+(bc+d^2)} = x$

on comparing coefficient of both side $(a^2 + bc)\,x + (ab + bd) = (ac + cd)\,x^2 + (bc + d^2)\,x$

$a^2 + bc = bc + d^2 \Rightarrow a = d$ or $a = -d$

and $ab + bd = 0 \Rightarrow b = 0$ or $a = -d$

and $ac + cd = 0 \Rightarrow c = 0$ or $a = -d$

which can be simultaneously true for $a = -d$

G-5. $h(x) = f(f(x)) = \begin{cases} f(x) & -1 \le f(x) \le 1 \\ f(x)^2 & 1 < f(x) \le 2 \end{cases} = \begin{cases} x & -1 \le x \le 1 \\ x^4 & 1 < x \le \sqrt{2} \end{cases}$

so domain of $h(x)$ is $[-1, \sqrt{2}\,]$. hence range of $h^{-1}(x)$ is $[-1, \sqrt{2}\,]$

$h(x) = f(f(x)) = \begin{cases} f(x) & -1 \le f(x) \le 1 \\ f(x)^2 & 1 < f(x) \le 2 \end{cases} = \begin{cases} x & -1 \le x \le 1 \\ x^4 & 1 < x \le \sqrt{2} \end{cases}$

G-6. $y = f(x)$ and $y = f^{-1}(x)$ can intersect at points other than $y = x$

e.g. $y = -x + c$ or $y = \sqrt{1-x^2}$

Section (H)

H-1. $f(x) = \sin^{-1}(|x-1| - 2)$. For domain $-1 \le |x-1| - 2 \le 1$

$\Rightarrow 1 \le |x-1| \le 3 \Rightarrow x - 1 \in [-3, -1] \cup [1, 3] \Rightarrow x \in [-2, 0] \cup [2, 4]$

H-2. $f(x) = \cot^{-1}\sqrt{x(x+3)} + \cos^{-1}\sqrt{x^2+3x+1}$

for domain

$x(x + 3) \ge 0$ and $0 \le x^2 + 3x + 1 \le 1$

$\Rightarrow x \in (-\infty, -3] \cup [0, \infty)$ and $x^2 + 3x + 1 \ge 0$ and $x^2 + 3x \le 0 \Rightarrow x \in [-3, 0]$

Taking intersection

$\therefore x \in \{-3, 0\}$

H-3. $\therefore -1 \le x \le 1$ ...(1)

$x \in R$ ...(2)

$x \le -1$ or $x \ge 1$ ...(3)

By $(1) \cap (2) \cap (3)$

$\Rightarrow x \in \{-1, 1\}$

H-4. Domain of $f(x)$ is $x \in \{-1, 1\}$ $f(-1) = -\dfrac{\pi}{2} - \dfrac{\pi}{4} + \pi = \dfrac{\pi}{4} \Rightarrow f(1) = \dfrac{\pi}{2} + \dfrac{\pi}{4} + 0 = \dfrac{3\pi}{4}$

H-5. $\operatorname{cosec}^{-1}(\cos x)$ is defined if $\cos x \ge 1$ or $\cos x \le -1 \Rightarrow \cos x = \pm 1 \Rightarrow x = n\pi$

H-6. $y = \sqrt{\sin^{-1}2x + \dfrac{\pi}{6}}$ For domain $\sin^{-1}2x + \dfrac{\pi}{6} \ge 0$

$\Rightarrow -\dfrac{\pi}{6} \le \sin^{-1}2x \le \dfrac{\pi}{2} \Rightarrow -\dfrac{1}{2} \le 2x \le 1 \Rightarrow -\dfrac{1}{4} \le x \le \dfrac{1}{2}$

H-7. $\sin^{-1}\left(\tan\dfrac{\pi}{4}\right) - \sin^{-1}\left(\sqrt{\dfrac{3}{x}}\right) - \dfrac{\pi}{6} = 0 \Rightarrow \sin^{-1}1 - \sin^{-1}\sqrt{\dfrac{3}{x}} = \dfrac{\pi}{6}$

$\Rightarrow \dfrac{\sqrt{3}}{2} = \sqrt{\dfrac{3}{x}} \Rightarrow \sqrt{x} = 2 \quad x = 4$

H-8. $4 - x^2 \ge 0$ & $-1 \le x^2 - 5 \le 1$

$x^2 \le 4$ & $4 \le x^2 - 5 \le 6$

So $x^2 = 4$ which satisfy the given equation

So $x = \pm 2$ number of solution = 2.

Section (I)

I-1. $\pi \le x \le 2\pi$ $\quad\quad$ $\cos^{-1}\cos x = 2\pi - x$

I-2. $\sin^{-1}x + \sin^{-1}y = \frac{\pi}{2} - \cos^{-1}x - \cos^{-1}y + \frac{\pi}{2} = \pi - (\cos^{-1}x + \cos^{-1}y) = \frac{2\pi}{3}$

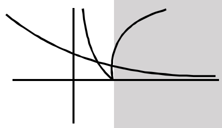
$\Rightarrow \cos^{-1}x + \cos^{-1}y = \frac{\pi}{3}$

I-3. $\theta = \sin^{-1}x + \cos^{-1}x - \tan^{-1}x = \frac{\pi}{2} - \tan^{-1}x$

Domain $x \in [-1, 1]$ $\quad\quad$ But given $x \ge 0$

$\Rightarrow x \in [0, 1] \quad \Rightarrow \quad \theta = \frac{\pi}{2} - \tan^{-1}x = \cot^{-1}x$

for $x \in [0, 1] \quad \Rightarrow \quad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}$



I-4.

$e^{-x} = |\ell nx|$

Section (J)

J-1. $= \cot\left(\frac{\pi}{2} + \sin^{-1}\frac{3}{5}\right) = -\tan\sin^{-1}\frac{3}{5} = \frac{-3}{4}$

J-2. $\tan^2(\sec^{-1}2) + \cot^2(\text{cosec}^{-1}3) \Rightarrow \tan^2(\tan^{-1}\sqrt{3}) + \cot^2(\cot^{-1}\sqrt{8}) \Rightarrow 3 + 8 = 11$

J-3. Let $f(x) = x^3 + 3x - \tan 2$

$f'(x) = 3x^2 + 3 > 0 \ \forall\ x \in R \Rightarrow f(x)$ is increasing and has exactly one real root

$f(0) = -\tan 2$ = positive

$f(-1) = -1 - 3 - \tan 2 = -4 - \tan 2$ = negative

$\therefore\ \alpha$ lies between $(-1,0)$

Now, $\cot^{-1}\alpha + \cot^{-1}\frac{1}{\alpha} - \frac{\pi}{2} = \cot^{-1}\alpha + \pi + \tan^{-1}\alpha - \frac{\pi}{2} = \pi$

J-4. Here, $x \in [0,4]$

Now, we have

$\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \cos^{-1}\left(\frac{\sqrt{x}}{2}\right) + \tan^{-1}y = \frac{2\pi}{3} \Rightarrow y = \frac{1}{\sqrt{3}}$

$\therefore$ Maximum value of $(x^2 + y^2) = 16 + \frac{1}{3} = \frac{49}{3}$ and minimum value of $(x^2 + y^2) = (0)^2 + \frac{1}{3} = \frac{1}{3}$

J-5. By property if $x < 0$ $\quad\quad$ $\tan^{-1}\frac{1}{x} = \cot^{-1}x - \pi$

$\therefore\ \tan^{-1}x + \tan^{-1}\frac{1}{x} = \tan^{-1}x + \cot^{-1}x - \pi = \frac{\pi}{2} - \pi \quad\quad \Rightarrow \tan^{-1}x + \tan^{-1}\frac{1}{x} = -\frac{\pi}{2}$

J-6. $\sin^{-1}x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2} \Rightarrow \sin^{-1}x + \cos^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{2} \Rightarrow x = \frac{1}{\sqrt{5}}$

J-7. $\dfrac{\dfrac{2.\frac{1}{5}}{1-\frac{1}{25}}-1}{1+\dfrac{2.\frac{1}{5}}{1-\frac{1}{25}}.1} = \dfrac{-7}{17}$

Section (K)

K-1. $f(x) = \tan^{-1}\dfrac{\frac{x}{3}-\frac{x}{\sqrt{3}}}{1+\frac{x^2}{3\sqrt{3}}} = \tan^{-1}\dfrac{x}{3} - \tan^{-1}\dfrac{x}{\sqrt{3}}$; $0 \le x \le 3$.

Hence, $f(x) = \tan^{-1}\left(\dfrac{x}{2}\right)$ $\Rightarrow$ range of f(x) is $\left[0, \dfrac{\pi}{4}\right]$

K-2. Using properties

$\therefore$ $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$ $\Rightarrow$ $\dfrac{a}{x} = \dfrac{x}{b}$ $\Rightarrow$ $x = \sqrt{ab}$

statement-1 is true $\tan^{-1}\left(\dfrac{m}{n}\right) + \tan^{-1}\left(\dfrac{1-\frac{m}{n}}{1+\frac{m}{n}}\right) = \tan^{-1}\dfrac{m}{n} + \tan^{-1} 1 - \tan^{-1}\dfrac{m}{n} = \dfrac{\pi}{4}$.

K-3. Given that, $\cos^{-1} x - \cos^{-1}\dfrac{y}{2} = \alpha \Rightarrow \cos^{-1}\left(\dfrac{xy}{2} + \sqrt{1-x^2}\sqrt{1-\dfrac{y^2}{4}}\right) = \alpha$

$\Rightarrow$ $\dfrac{xy}{2} + \sqrt{1-x^2}\sqrt{1-\dfrac{y^2}{4}} = \cos\alpha$ $\Rightarrow$ $2\sqrt{1-x^2}\sqrt{1-\dfrac{y^2}{4}} = 2\cos\alpha - xy$

On squaring both sides, we get $\dfrac{4(1-x^2)(4-y^2)}{4} = 4\cos^2\alpha + x^2y^2 - 4xy\cos\alpha$

$\Rightarrow$ $4 - 4x^2 - y^2 + x^2y^2 = 4\cos^2\alpha + x^2y^2 - 4xy\cos\alpha$

$\Rightarrow$ $4x^2 - 4xy\cos\alpha + y^2 = 4\sin^2\alpha$

PART - III

1. Obvious (according diffenations of Reflexive, Symmectric and Transitive relation)

2. **(A)** $f(\Delta)$ = Area of Δ. Different triangles can have same area.

$\therefore$ Many on function. Area of triangle is positive hence onto function

(B) $f(x) = \cot^{-1}(2x - x^2 - 2) = \cot^{-1}(-1 - (x-1)^2)$

$\Rightarrow$ $-1 - (x-1)^2 \le -1$

$\Rightarrow$ f(0) = f(2). Hence f(x) is many-one.

$\Rightarrow$ $\cot^{-1}(2x - x^2 - 2) \in \left[\dfrac{3\pi}{4}, \pi\right)$

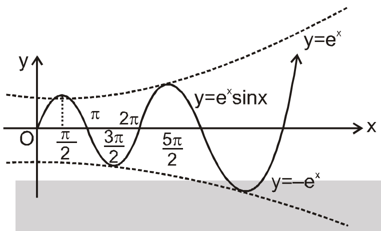
Hence, f(x) is onto. Also f(3) = f(–1), hence function is many-one.

$\Rightarrow$ $-1-(x-1)^2=-5.$

(C) $f(x)=\dfrac{2x^2-x+1}{(7x^2-4x+4)}$ $\Rightarrow$ $f'(x)=\dfrac{-x^2+2x}{(7x^2-4x+4)^2}\Rightarrow$ f(x) is not monotonic

$\therefore$ f(x) is many one.

(D)



Clearly, from the graph that f(x) is many-one and onto.

3. Obvious

4. (A) $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\dfrac{\pi}{2}$ $\Rightarrow$ $x\in[0,1]$

(B) $\sin^{-1}\sqrt{x}+\cos^{-1}\left(1-\sqrt{x^2}\right)=0$ $\Rightarrow$ $\cos^{-1}\sqrt{1-x^2}=-\sin^{-1}(x)\Rightarrow$ $x\in[-1,0]$

(C) $g\left(\dfrac{1-x^2}{1+x^2}\right)=2h(x)$ $\Rightarrow$ $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)=2\tan^{-1}x$ $\Rightarrow$ $x\in[0,\infty)$

(D) $h(x)+h(1)=h\left(\dfrac{1+x}{1-x}\right)$ $\Rightarrow$ $\tan^{-1}x+\tan^{-1}=\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$ $\Rightarrow$ $x\in(-\infty,1)$

5. (A) Let $x=\sqrt{\dfrac{a(a+b+c)}{bc}}$, $y=\sqrt{\dfrac{b(a+b+c)}{ac}}$, $z=\sqrt{\dfrac{c(a+b+c)}{ab}}$, $x,y,z>0$

$\theta=\tan^{-1}x+\tan^{-1}y+\tan^{-1}z$

Now $x+y+z=\sqrt{\dfrac{a(a+b+c)}{bc}}+\sqrt{\dfrac{b(a+b+c)}{ac}}+\sqrt{\dfrac{c(a+b+c)}{ab}}=\dfrac{(a+b+c)^{3/2}}{\sqrt{abc}}$

and $xyz=\dfrac{(a+b+c)^{3/2}}{\sqrt{abc}}$ $\Rightarrow$ $x+y+z=xyz$ $\Rightarrow\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$

Hence $\theta=\pi$

(B) Let $\alpha=\tan^{-1}(\cot A)$

$\Rightarrow\beta=\tan^{-1}(\cot^3 A)$

$\Rightarrow\tan(\alpha+\beta)=\dfrac{\cot A+\cot^3 A}{1-\cot^4 A}$

R.H.S. is negative $\Rightarrow\pi<\alpha+\beta<\dfrac{\pi}{2}$

$\Rightarrow$ $\tan(\alpha+\beta-\pi)=\dfrac{\cot A}{1-\cot^2 A}=-\dfrac{\tan 2A}{2}$

$\Rightarrow \quad \alpha + \beta = \pi - \tan^{-1}\left(\dfrac{\tan 2A}{2}\right)$ G.E. = π independent of A.

(C) If $x < 0$, then $\dfrac{1}{2}\{\cos^{-1}(2x^2 - 1) + 2\cos^{-1} x\} \Rightarrow x = \cos\theta,\ \pi/2 < \theta < \pi$

$\dfrac{1}{2}\{\cos^{-1}(2x^2 - 1) + 2\cos^{-1} x\} = \dfrac{1}{2}\{\cos^{-1}(\cos 2\theta) + 2\cos^{-1}x\}$

$= \dfrac{1}{2}\{\cos^{-1}(\cos 2\theta) + 2\cos^{-1}x\} = \dfrac{1}{2}\{-2\theta + 2\pi + 2\theta\} = \pi$

(D) $\sin^{-1}\left(\dfrac{3}{5}\right) - \cos^{-1}\left(\dfrac{12}{13}\right) + \cos^{-1}\left(\dfrac{16}{65}\right) = \sin^{-1}\left(\dfrac{3}{5}\right) - \sin^{-1}\left(\dfrac{5}{13}\right) + \cos^{-1}\left(\dfrac{16}{65}\right)$

$= \sin^{-1}\left\{\dfrac{3}{5}\cdot\sqrt{1-\left(\dfrac{5}{13}\right)^2} - \dfrac{5}{13}\sqrt{1-\left(\dfrac{3}{5}\right)^2}\right\} + \cos^{-1}\left(\dfrac{16}{65}\right)$

$= \sin^{-1}\left\{\dfrac{3}{5}\cdot\dfrac{12}{13} - \dfrac{5}{13}\cdot\dfrac{4}{5}\right\} + \cos^{-1}\left(\dfrac{16}{65}\right) = \sin^{-1}\left(\dfrac{16}{65}\right) + \cos^{-1}\left(\dfrac{16}{65}\right) = \dfrac{\pi}{2}$

EXERCISE # 2

PART - I

1. For any $x \in R$, we have $x - x + \sqrt{2} = \sqrt{2}$ an irrational number

$\Rightarrow$ xRx for all x. So, R is reflexive.

R is not symmetric, because $\sqrt{2}$R1 but $1 \not R \sqrt{2}$, R is not transitive also because $\sqrt{2}$ R1 and $\sqrt{2}$ 1R2 $\sqrt{2}$ but $\sqrt{2} \not R\, 2\sqrt{2}$

2. $((m,n), (p,q)) \in S$

$\Rightarrow m + q = n + p$

$((p,q), (r,s)) \in S \qquad \Rightarrow p + s = r + q$

$\Rightarrow ((r,s), (m,n)) \in s$

as $r + n = m + s$

Now if we add above equation

$((m,n), (p,q)) \in s \Rightarrow m + q = n + p$

$\Rightarrow (n+p) = m + q$ & hence $((P,q), (m,n))$

3. $R_1 : m + 4n = 5n + (m - n)$

$R_2 : m + 9n = 10n + (m - n)$

If $5n + (m - n)$ is divisible by 5 then $10n + (m - n)$ is also divisible by 5 and vice versa.

hence $R_1 = R_2$

Also R_1 & R_2 is symmetric relation on Z.

4. $(x, y) \in X \quad \Leftrightarrow \quad x < y$ and $y = x + 5 \quad \Rightarrow \quad (x, x) \notin X$, Hence not reflective

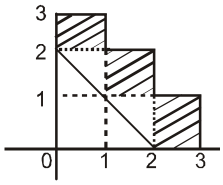
If $(x, y) \in X \quad \Rightarrow \quad x < y$ and $y = x + 5$

$(y, x) \in X \quad \Rightarrow$ X is not symmetric

Let $(x, y) \in X$ and $(y, z) \in X \quad \Rightarrow \quad x < y;\ y = x + 5$ and $y < z;\ z = y + 5$

$\Rightarrow x < z$ and $z = x + 10 \quad \Rightarrow \quad (x, z) \notin X$

Not transitive

5. 

$1 + 1 + 1 = 3$

6. $\left[x-\frac{1}{3}\right]=\left[x+\frac{2}{3}\right]-1$

$\therefore$ the equation becomes $[x] + \left[x+\frac{1}{2}\right] + \left[x+\frac{2}{3}\right] = 9$

Let $n \le x < n + 1$, then $[x] = n$, $\left[x+\frac{1}{2}\right] = n$ or $n + 1$, $\left[x+\frac{2}{3}\right] = n$ or $n + 1$

$\therefore$ $[x] + \left[x+\frac{1}{2}\right] + \left[x+\frac{2}{3}\right] = 3n, 3n + 1, 3n + 2$

$\therefore$ the only possible case is $3n = 9$ i.e. $n = 3$

$\therefore$ $3 \le x < 4$, $3 \le x + \frac{1}{2} < 4$ and $3 \le x + \frac{2}{3} < 4$ i.e. $3 \le x < \frac{10}{3}$

$\therefore$ $a = 3$, $b = \frac{10}{3}$

7. $f(x) = \{x\} \times 100$

$f(\sqrt{3}) = \{\sqrt{3}\} \times 100$

$[f(\sqrt{3})] = [73.2\text{-------}] = 73$

8. $x^2 - 10x + 25\text{sgn}(x^2 + 4x - 32) \le 0 \Rightarrow$

$$\begin{cases} x^2-10x+25\le 0 & \text{when } (x+8)(x-4)>0 \text{ or } x\in(-\infty, -8)\cup(4, \infty) \\ x^2-10x-25\le 0 & \text{when } (x+8)(x-4)<0 \text{ or } x\in(-8, 4) \\ x^2-10x\le 0 & \text{when } (x+8)(x-4)=0 \text{ or } x\in\{4, -8\} \end{cases}$$

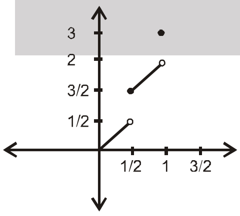
total solutions $= -2, -1, 0, 1, 2, 3, 4, 5$

9. $[x + [2x]] < 3$

$[x] + [2x] < 3$

True for

$x < 1$



10. (i) $-x^2 + 5x - 6 \ge 0 \Rightarrow$ $x \in [2, 3]$

and

(ii) $2\{x\} < 1$ $\Rightarrow$ $\{x\} < \frac{1}{2}$

$\Rightarrow$ $x \in \left[2, \frac{5}{2}\right) \cup \{3\}$

11. $f(x) = \log_{1/2}\left(-\log_2\left(1+\frac{1}{\sqrt[4]{x}}\right)-1\right) \Rightarrow -\log_2\left(1+\frac{1}{x^{1/4}}\right) - 1 > 0$

$\Rightarrow \quad -\infty < \log_2\left(1+\frac{1}{x^{1/4}}\right) < -1 \Rightarrow 0 < 1 + \frac{1}{x^{1/4}} < \frac{1}{2} \Rightarrow -1 < \frac{1}{x^{1/4}} < -\frac{1}{2}$

$\Rightarrow \quad x \in \phi$ (null set) $\Rightarrow \quad x \in \phi$

12. $q^2 - 4pr = 0, p > 0 \quad \Rightarrow \quad f(x) = \log(px^3 + (p+q)x^2 + (q+r)x + r)$

Let $g(x) = px^3 + (p+q)x^2 + (q+r)x + r \quad \Rightarrow \quad g(x) = (x+1)(px^2 + qx + r)$

Discriminant of $px^2 + qx + r$ is $q^2 - 4pr = 0$

Domain $(x+1)(px^2+qx+r) > 0 \quad \Rightarrow \quad p(x+1)\left(x+\frac{q}{2p}\right)^2 > 0$

$\Rightarrow \quad x \neq -\frac{q}{2p}$ and $x > -1 \quad \therefore \quad x \in R - [(-\infty, -1] \cup \left\{-\frac{q}{2p}\right\}$

13. $f(x) = \frac{x-[x]}{1+x-[x]} = \frac{\{x\}}{1+\{x\}} = 1 - \frac{1}{1+\{x\}}$

$\because \quad \{x\} \in [0, 1) \quad \Rightarrow \quad f(x) \in \left[0, \frac{1}{2}\right)$

14. $f(x) = \frac{e^x - e^{-|x|}}{e^x + e^{|x|}}$

if $x \geq 0$, $f(x) = \frac{e^x - e^{-x}}{2e^x} = \frac{1}{2} - \frac{1}{2(e^x)^2} = \frac{1}{2}\left(1 - \frac{1}{(e^x)^2}\right)$; $f(x) \in \left[0, \frac{1}{2}\right)$ ....(i)

$f(x) \in \left[0, \frac{1}{2}\right)$

if $x < 0$, $f(x) = \frac{e^x - e^x}{e^x + e^{-x}} = 0$ ....(ii)

$\therefore$ range of $f(x)$ is (i) $\cup$ (ii) $= \left[0, \frac{1}{2}\right)$

15. Here $(2 - \log_2(16\sin^2x + 1)) > 0 \quad \Rightarrow \quad 0 < 16\sin^2x + 1 < 4$

$\Rightarrow \quad \frac{-1}{16} \leq \sin^2x < \frac{3}{16} \quad \Rightarrow \quad 1 \leq 16\sin^2x + 1 \leq 4 \Rightarrow \quad 0 \leq \log_2(16\sin^2x + 1) < 2$

$\Rightarrow \quad 2 \geq 2 - \log_2(16\sin^2x + 1) > 0$

$\Rightarrow \quad \log_{\sqrt{2}} 2 \geq \log_{\sqrt{2}}(2 - \log_2(16\sin^2x + 1)) > -\infty \quad \Rightarrow \quad 2 \geq y > -\infty$

Hence range is $y \in (-\infty, 2]$

16. (A) $\sqrt{1+\sin x} = \left|\sin\frac{x}{2} + \cos\frac{x}{2}\right|$

$\sin\frac{x}{2} + \cos\frac{x}{2}$ non-identical function

(B) $x, \frac{x^2}{x}$ Non-identical function

(C) $\sqrt{x^2} = |x|, x \in R$

$\left(\sqrt{x}\right)^2 = x, x \in R^+ \cup \{0\}$ non-identical function

(D) $\ell nx^3 + \ell n x^2 = 5\,\ell nx, x > 0$

$5\ell x, x > 0$ identical function

17. $f(6\{x\}^2 - 5\{x\} + 1) \Rightarrow f((3\{x\} - 1)(2\{x\} - 1)) \Rightarrow (3\{x\} - 1)(2\{x\} - 1) \le 0$

or $\{x\} \in \left[\frac{1}{3}, \frac{1}{2}\right]$ $\therefore$ $x \in \bigcup\limits_{n \in I} \left[n + \frac{1}{3}, n + \frac{1}{2}\right]$

18. $f : (e, \infty) \to R \Rightarrow f(x) = \ell n(\ell n(\ell n\ x))$

$D : \ell n(\ell n\ x) > 0$ or $\ell nx > 1$ or $x > e$

$R : (-\infty, \infty) \Rightarrow$ one-one and onto function

19. $f(x) = 2[x] + \cos x$; $f(x) = \cos x \quad x \in [0, 1)$

$= 2 + \cos x \quad x \in [1, 2)$

$= 4 + \cos x \quad x \in [2, 3)$

$= 6 + \cos x \quad x \in [3, 4)$

for $x \in [0, 1)$ $f'(x) = -ve$

$x \in [1, 2)$ $f'(x) = -ve$

$x \in [2, 3)$ $f'(x) = -ve$

$x \in [3, 4)$ $f'(x) = +ve$

$\Rightarrow$ function is not one-one

if $x \in [0, 1)$ range : $[1, \cos 1)$

$x \in [1, 2)$ range : $[2 + \cos 1, 2 + \cos 2)$

not onto function

20. $f(2) = f(3^{1/4}) = 0$ many one

$f(x) \ne -\sqrt{3}$ $\therefore$ Into

21. $f(x) = |x - 1|$ $f : R^+ \to R$; $g(x) = e^x$, $g : [-1, \infty) \to R$

$fog(x) = f[g(x)] = |e^x - 1|$

$D : [-1, \infty)$; $R : [0, \infty)$

22. $x \in (2, 4) \Rightarrow \left[\frac{x}{2}\right] = 1$

so $\Rightarrow f(x) = x - 1 \Rightarrow y = x - 1$

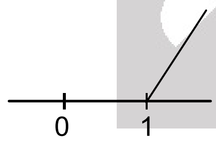
$\Rightarrow x = y + 1 \Rightarrow f^{-1}(x) = x + 1$

23. $f : R \to R$, $f(x) = x^3 + ax^2 + bx + c$

$f'(x) = 3x^2 + 2ax + b$

$D \le 0$ or $4a^2 - 12b \le 0$ or $a^2 \le 3b4$

24.



$f : [1, \infty) \to [1, \infty)$

$f(x) = 2^{x(x-1)}$

$x(x - 1)$ is strictly increasing in domain

$f(x) = 2^{x(x-1)}$ is one one & onto function so inverse is defined

$2^{x(x-1)} = y \Rightarrow x^2 - x = \log_2 y \Rightarrow x^2 - x - \log_2 y = 0$

$$x = \left(\frac{1 \pm \sqrt{1 + 4\log_2 y}}{2}\right)$$

–ve sign rejected as domain range of f^n is $[1, \infty)$

$$f^{-1}(x) = \left(\frac{1 \pm \sqrt{1 + 4\log_2 x}}{2}\right)$$

25. $f : N \to N$

$f(x) = x + (-1)^{x-1} \Rightarrow f(x) = \begin{cases} x-1, & x \to \text{Even natural} \\ x+1, & x \to \text{Odd natural} \end{cases} \Rightarrow f^{-1}(x) = \begin{cases} x+1, & x \to \text{Odd natural} \\ x-1, & x \to \text{Even natural} \end{cases}$

$\therefore \quad f^{-1}(x) = x + (-1)^{x-1}$

26. $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right) \quad x \neq 0$

let $\theta = \frac{1}{2}\cos^{-1}x$, $2\theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$= \tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right) = \frac{1+\tan\theta}{1-\tan\theta}+\frac{1-\tan\theta}{1+\tan\theta} = 2\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right) = \frac{2}{\cos 2\theta} = \frac{2}{\cos\cos^{-1}x} = \frac{2}{x}$

27. $\cot^{-1}\left\{\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right\} \quad \frac{\pi}{2} < x < \pi$

Rationalize the term in the bracket

$= \cot^{-1}\left(\frac{2+2\sqrt{1-\sin^2 x}}{-2\sin x}\right) = \cot^{-1}\left(\frac{1-\cos x}{-\sin x}\right) = \cot^{-1}\left(-\tan\frac{x}{2}\right) = \frac{\pi}{2}-\tan^{-1}\left(-\tan\frac{x}{2}\right) = \frac{\pi}{2}+\tan^{-1}\tan\frac{x}{2}$

since $\frac{x}{2} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \quad = \frac{\pi}{2}+\frac{x}{2}$

28. $f(x) = \sin^{-1}\left(\frac{1+x^3}{2x^{3/2}}\right) + \sqrt{\sin(\sin x)} + \log_{(3\{x\}+1)}(x^2+1)$

Domain : $3\{x\} + 1 \neq 1$ or $0 \Rightarrow x \notin I$ and $-1 \le \frac{1+x^3}{2x^{3/2}} \le 1$

$\Rightarrow -2x^{3/2} \le 1 + x^3 \le 2x^{3/2} \qquad \Rightarrow 1 + x^3 + 2x^{3/2} \ge 0 \qquad \Rightarrow (1 + x^{3/2})^2 \ge 0$

$\Rightarrow x \in R \qquad \Rightarrow 1 + x^3 - 2x^{3/2} \le 0$

or $(1 - x^{3/2})^2 \le 0$ or $1 - x^{3/2} = 0$ or $x = 1$

Hence domain $x \in \phi$

29. Clearly (B) also satisfies (i), (ii), (iii) but not (iv) but (A) satisfies all the condition

30. $([\cot^{-1}x] - 3)^2 \le 0 \qquad \Rightarrow [\cot^{-1}x] = 3$

$\Rightarrow 3 \le \cot^{-1}x < 4 \qquad \Rightarrow 3 \le \cot^{-1}x < \pi < 4$

$\Rightarrow -\infty < x \le \cot 3$

31. $\sin^{-1}\sin 5 = 5 - 2\pi \qquad \Rightarrow x^2 - 4x - 5 + 2\pi < 0 \Rightarrow \left(x-\left(2-\sqrt{9-2\pi}\right)\right)\left(x-\left(2+\sqrt{9-2\pi}\right)\right) < 0$

$\Rightarrow x \in \left(2-\sqrt{9-2\pi}, 2+\sqrt{9-2\pi}\right)$

32. $\sin^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}.......\right) + \cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}+...\right) = \Rightarrow \sin^{-1}\left(\frac{x}{1+(x/2)}\right) + \cos^{-1}\left(\frac{x^2}{1+(x^2/2)}\right) = \frac{\pi}{2}$

$\frac{2x}{2+x} = \frac{2x^2}{2+x^2} \Rightarrow \qquad 2x + x^3 = 2x^2 + x^3$

$x = 0, 1 \qquad$ But $\quad \therefore |x| > 0$

so $x = 1$ is the only answer.

33. Given that $\cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha}) = x$ .....(i)

We know that, $\cot^{-1}(\sqrt{\cos\alpha}) + \tan^{-1}(\sqrt{\cos\alpha}) = \frac{\pi}{2}$ ......(ii)

On adding equations (i) and (ii), we get

$$2\cot^{-1}(\sqrt{\cos\alpha}) = \frac{\pi}{2} + x \Rightarrow \sqrt{\cos\alpha} = \cot\left(\frac{\pi}{4}+\frac{x}{2}\right) \Rightarrow \sqrt{\cos\alpha} = \frac{\cot\frac{x}{2}-1}{1+\cot\frac{x}{2}}$$

$$\Rightarrow \sqrt{\cos\alpha} = \frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}} \Rightarrow \cos\alpha = \frac{1-\sin x}{1+\sin x} \Rightarrow \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{1-\sin x}{1+\sin x}$$

Applying componendo and dividendo rule, we get $\sin x = \tan^2\left(\frac{\alpha}{2}\right)$

34. Given that, $\sin^{-1} x = 2\sin^{-1}\alpha$

Since, $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \le 2\sin^{-1}\alpha \le \frac{\pi}{2}$

$$\Rightarrow -\frac{\pi}{4} \le \sin^{-1}\alpha \le \frac{\pi}{4} \Rightarrow \sin\left(-\frac{\pi}{4}\right) \le \alpha \le \sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow -\frac{1}{\sqrt{2}} \le \alpha \le \frac{1}{\sqrt{2}} \Rightarrow |\alpha| \le \frac{1}{\sqrt{2}}$$

35. $f(x) = \cot^{-1}x \quad R^+ \to \left(0, \frac{\pi}{2}\right)$; $g(x) = 2x - x^2 \quad R \to R$

$f(g(x)) = \cot^{-1}(2x - x^2)$, where $x \in (0, 2)$ & $2x - x^2 \in (0, 1]$ hence $f(g(x)) \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$

36. $f(x) = e^{\cos^{-1}\left(\sin\left(x+\frac{\pi}{3}\right)\right)}$

Domain $-1 \le \sin\left(x+\frac{\pi}{3}\right) \le 1 + \frac{\pi}{2} \le x + \frac{\pi}{3} \le \frac{3\pi}{2} \Rightarrow \frac{\pi}{6} \le x \le \frac{7\pi}{6}$

$g(x) = \operatorname{cosec}^{-1}\left(\frac{4-2\cos x}{3}\right) \Rightarrow \frac{4-2\cos x}{3} \ge 1$

or $\cos x \le \frac{1}{2} \Rightarrow x \in \left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$

Domain of h(x) : $x \in \left[\frac{\pi}{3}, \frac{7\pi}{6}\right] \Rightarrow h(x) = e^{\cos^{-1}\left(\sin\left(x+\frac{\pi}{3}\right)\right)} = e^{\cos^{-1}\left[-1, \frac{\sqrt{3}}{2}\right]}$ range of h(x) : $[e^{\pi/6}, e^{\pi}]$

PART–II

1. Let $x = I + f$, $f \in (0, 1)$ (as $f \ne 0$)

so $f, I, I + f$ are in H.P.

i.e. $I = \frac{2f(I+f)}{f+I+f} \Rightarrow I^2 + 2If = 2If + 2f^2$

I $\quad I = \pm f\sqrt{2}$

If $\quad I = 1,\ f = \dfrac{1}{\sqrt{2}} \quad \Rightarrow \quad x = \dfrac{\sqrt{2}+1}{\sqrt{2}}$

If $\quad I = -1,\ f = +\dfrac{1}{\sqrt{2}} \Rightarrow \quad x = \dfrac{-\sqrt{2}+1}{\sqrt{2}}$

so $|\alpha\beta| = \left|\dfrac{(\sqrt{2}+1)(-\sqrt{2}+1)}{2}\right| = \dfrac{1}{2} \qquad \Rightarrow 49|\alpha\beta| = 24.50$

2. $7x - x^2 - 6 \ge 0 \quad \Rightarrow \quad x^2 - 7(x) + 6 \le 0$

$(x-1)(x-6) \le 0 \quad \Rightarrow \quad x \in [1, 6]$ and $\sin x + \cos x \ge 0 \quad \Rightarrow \quad \left[1, \dfrac{3\pi}{4}\right] \cup \left[\dfrac{7\pi}{4}, 6\right]$

3. $\left[x + \dfrac{1}{2}\right] = 2 \quad \Rightarrow \quad 2 \le x + \dfrac{1}{2} < 3 \quad \Rightarrow \quad \dfrac{3}{2} \le x < \dfrac{5}{2}$

4. $f(x) = \dfrac{(x^2+x+2)(x+1)}{(x^2+x+1)(x+1)};x \in R - \{0\}$

$f(x) = \dfrac{x^2+x+2}{x^2+x+1};x \in R - \{0, -1\}$

$y = \dfrac{x^2+x+2}{x^2+x+1} \Rightarrow (y-1)x^2 + (y-1)x + y - 2 = 0$

$y \ne 1, \quad D \ge 0$

$(y-1)^2 - 4(y-1)(y-2) = 0 \Rightarrow 1 < y \le 7/3$ at $\quad x = 0$ we get $y = 2$

$x = 0\ y = 2$

& $y = 2 \Rightarrow 2 = \dfrac{x^2+x+2}{x^2+x+1} \Rightarrow x(x+1) = 0 \Rightarrow x = 0, -1$ but $x \ne 0, -1$ so $y \ne 2$ Range $\left(1, \dfrac{7}{3}\right] - \{2\}$.

5.
$$f(x) = \begin{cases} |\sin 2x| & 0 \le x < \dfrac{\pi}{2} \quad \Rightarrow \quad 0 \le 2x < \pi \quad \Rightarrow \quad 0 \le \sin 2x \le 1 \\ 0 & \dfrac{\pi}{2} \le x < \pi \\ |-\sin 2x| & \pi \le x < \dfrac{3\pi}{2} \quad \Rightarrow \quad 2\pi \le 2x < 3\pi \quad \Rightarrow \quad \theta \le |-\sin 2x| \le 1 \\ 0 & \dfrac{3\pi}{2} \le x \le 2\pi \end{cases}$$

so range [0, 1].

6. f & g are 2 distinct functions $[-1, 1] \to [0, 2]$ onto functions

So f & g are either $-x+1$ or $x+1$

Case-I: $f(x) = -x + 1 \quad ; \quad g(x) = x + 1$

$h(x) = \dfrac{f(x)}{g(x)} = \dfrac{1-x}{1+x} \quad ; \quad h(h(x)) = \dfrac{1 - \dfrac{1-x}{1+x}}{1 + \dfrac{1-x}{1+x}} = x$

$h(1/x) = \dfrac{x-1}{x+1} \quad ; \quad h(h(1/x)] = \dfrac{1 - \dfrac{x-1}{x+1}}{1 + \dfrac{x-1}{x+1}} = \dfrac{1}{x}$

$\left|h(h(x)) + h\left(h\left(\frac{1}{x}\right)\right)\right|$ = |x + 1/x| > 2 as domain does not contain point x = ± 1

Case-II: f(x) = 1 + x ; g(x) = 1 – x

$h(x) = \frac{1+x}{1-x}$; $h(h(x)) = \frac{1+\frac{1+x}{1-x}}{1-\frac{1+x}{1-x}} = -\frac{1}{x}$

h(h(1/x)) = – x ; $\left|h(h(x)) + h\left(h\left(\frac{1}{x}\right)\right)\right|$ = |–x – 1/x| = (x + 1/x) > 2

7. $g(x) = \frac{x-1}{x}$ and h(x) = x ⇒ f(x) . g(x). h(x) = – 1.

8. $f(-x) = -[ax^7 + bx^3 + cx] - 5$; $f(-x) = -[f(x) + 5] - 5$

f(–x) = – f(x) = – 10 put x = 7 (x = 7)

f(7) = – 17 so f(7) + 17 cosx = – 17 (cos x – 1) ∈ [–34, 0]

9. $f(x) = \frac{4a-7}{3}x^3 + (a-3)x^2 + x + 5$ ⇒ $f'(x) = (4a-7)x^2 + 2(a-3)x + 1$

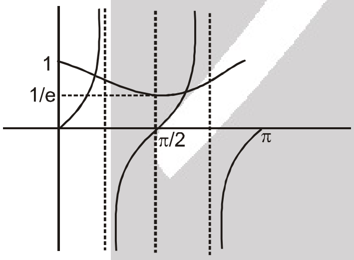
D ≤ 0 for all x ∈ R ⇒ x ∈ R D ≤ 0

$4(a-3)^2 - 4(4a-7) \le 0$ ⇒ $a^2 + 9 - 6a - 4a + 7 \le 0$

$a^2 - 10a + 16 \le 0$ ⇒ $(a-8)(a-2) \le 0$

or a ∈ [2, 8] ⇒ f′(x) is always +ve for a ∈ [2, 8]

10.



Period of $e^{-\sin^2 x}$ is π

and that of tan2x is π/2

so number of solutions in (0, π) is 2

Number of solutions in [0, π] is 2

so number of solution in [0, 10π] = 20

11. $f(x) = ([a]^2 - 5[a] + 4)x^3 - (6\{a\}^2 - 5\{a\} + 1)x - \tan x(\text{sgn } x)$

x > 0

$f(x) = ([a]^2 - 5[a] + 4)x^3 - (6\{a\}^2 - 5\{a\} + 1)x - \tan x$

x < 0

$f(x) = ([a]^2 - 5[a] + 4)x^3 - (6\{a\}^2 - 5\{a\} + 1)x + \tan x$

Given that function is even function $\forall\ x \in R$

So $\quad f(x) - f(-x) = 0\ \forall\ n \in R$

$2x^3([a]^2 - 5[a] + 4) - 2x\,(6\{a\}^2 - 5\{a\} + 1) = 0$

So this equation should be independent from x

$\therefore$ coff. of x^3 & x will be zero.

$[a]^2 - 5[a] + 4 = 0 \quad ; \quad 6\{a\}^2 - 5\{a\} + 1 = 0$

$[a] = 1, 4 \quad ; \quad \{a\} = 1/2, 1/3$

$a = 1 + 1/2,\ 4 + 1/2,\ 1 + 1/3,\ 4 + 1/3 = 3/2,\ 9/2,\ 4/3,\ 13/3$

Sum $= 3/2 + 9/2 + 4/3 + 13/3 = 6 + 17/3 = 35/3$

12. 21 x
22 y
23 z

	case-I	**case-II**	**case-III**
$f(21) = x$	T	F	F
$f(22) \neq x$	F	T	F
$f(23) \neq y$	F	F	T

case-I $\quad f(22) = x,\ f(23) = y$

then $f(21) = x$ is not true

case-II $\quad f(23) = y,\ f(22) = z,\ f(21) = x$

not possible

case-III $\quad f(22) = x,\ f(23) = z,\ f(21) = y$

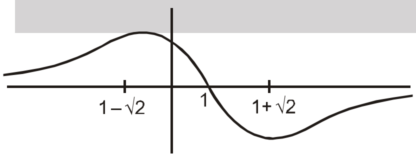
$\therefore \quad f^{-1}(x) = 22$

13. $f(x) = \dfrac{1-x}{1+x^2} \quad \Rightarrow \quad f'(x) = 0 \quad$ at $x = 1 \pm \sqrt{2}$

for $x \in \left[-\sqrt{2}+1,\ 1+\sqrt{2}\right]$ f is bijective function hence f is invertible.

$\dfrac{1-x}{1+x^2} = y$

or $\quad x^2y + x + (y - 1) = 0$



or $\quad x = \dfrac{-1 \pm \sqrt{1 - 4y(y-1)}}{2y} = \dfrac{-1 \pm \sqrt{4y - 4y^2 + 1}}{2y}$

$$f^{-1}(x) = \begin{cases} \dfrac{-1 + \sqrt{4x - 4x^2 + 1}}{2x}, & x \neq 0 \\ 1 \quad , & x = 0 \end{cases} \qquad \text{as } f(1) = 0$$

14. $\frac{x^3+1}{2} = 2\sqrt[3]{2x-1}$

Let $f(x) = \frac{x^3+1}{2}$ $\Rightarrow$ $f^{-1}(x) = \sqrt[3]{2x-1}$

Equation becomes $f(x) = f^{-1}(x)$

$\Rightarrow$ $f(x) = x$ $\Rightarrow$ $\frac{x^3+1}{2} = x$ $\Rightarrow$ $x^3 - 2x + 1 = 0$

$\Rightarrow$ $(x-1)(x^2+x-1) = 0$ $\Rightarrow$ $x = 1, \frac{-1\pm\sqrt{5}}{2}$

Allter :

Let $y = \sqrt[3]{2x-1}$ $\Rightarrow$ $y^3 - 2x + 1 = 0$ and $x^3 - 2y + 1 = 0$

$\Rightarrow$ $(y^3 - 2x + 1) - (x^3 - 2y + 1) = 0$ $\Rightarrow$ $(y - x)(y^2 + xy + x^2 + 2) = 0$

$\Rightarrow$ $y = x$ or $y^2 + xy + x^2 + 2 = 0$ $\Rightarrow$ $y = x$ or $(x + y)^2 + x^2 + y^2 + 4 = 0$

Putting $y = x$ in $y = \sqrt[3]{2x-1}$, we get $y = x$

$x^3 - 2x + 1 = 0$

Which yields the values $x = 1, \frac{-1\pm\sqrt{5}}{2}$ $x = 1, \frac{-1\pm\sqrt{5}}{2}$

15. $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$-1 \le x, y, z \le 1$

let $x = \cos A$, $y = \cos B$, $z = \cos C$

where $0 \le A, B, C \le \pi$

$A + B + C = \pi$

$x^2 + y^2 + z^2 + 2xyz = \cos^2A + \cos^2B + \cos^2C + 2\cos A\cos B\cos C$

$= \sum\left(\frac{1+\cos 2A}{2}\right) + 2\cos A\cos B\cos C = \frac{3}{2} + \frac{1}{2}(-1 - 4\cos A\cos B\cos C) + 2\cos A\cos B\cos C = 1$

16. **Case-I :** $x \ge 0$

Let $\cot^{-1}x = \theta$

$\therefore$ $\theta \in \left(0, \frac{\pi}{2}\right]$ $\Rightarrow$ $x = \cot\theta$

$\therefore$ $\sin\theta = \frac{1}{\sqrt{1+x^2}}$ $\Rightarrow$ $\sin^{-1}\sin\theta = \sin^{-1}\frac{1}{\sqrt{1+x^2}}$ $\Rightarrow$ $\theta = \sin^{-1}\frac{1}{\sqrt{1+x^2}}$

Case-II : $x < 0$

Let $\cot^{-1}x = \theta$

$\therefore$ $\theta \in \left(\frac{\pi}{2}, \pi\right)$ $\Rightarrow$ $\cot\theta = x$

$\therefore$ $\sin\theta = \frac{1}{\sqrt{1+x^2}}$ $\Rightarrow$ $\sin^{-1}\sin\theta = \sin^{-1}\frac{1}{\sqrt{1+x^2}}$ $\Rightarrow$ $\pi - \theta = \sin^{-1}\frac{1}{\sqrt{1+x^2}}$

$\Rightarrow$ $\theta = \pi - \sin^{-1}\frac{1}{\sqrt{1+x^2}}$

Therefore,

LHS = $\begin{cases} \cos\tan^{-1}\sin\sin^{-1}\dfrac{1}{\sqrt{1+x^2}} & \text{, if } x \ge 0 \\ \cos\tan^{-1}\sin\left(\pi - \sin^{-1}\dfrac{1}{\sqrt{1+x^2}}\right) & \text{, if } x < 0 \end{cases}$ = $\cos\tan^{-1}\sin\sin^{-1}\dfrac{1}{\sqrt{1+x^2}}$; $x \in R$ = $\cos\tan^{-1}\dfrac{1}{\sqrt{1+x^2}}$

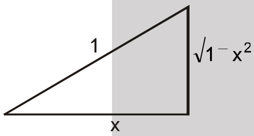
Let $\phi = \tan^{-1}\dfrac{1}{\sqrt{1+x^2}}$

As $\dfrac{1}{\sqrt{1+x^2}} \in (0, 1]$ $\quad \therefore\ \phi \in \left(0, \dfrac{\pi}{4}\right]$ $\quad \therefore\ \tan\phi = \dfrac{1}{\sqrt{1+x^2}}$ $\quad \therefore\ \cos\phi = \sqrt{\dfrac{1+x^2}{2+x^2}}$

17. $\cot^{-1}\dfrac{n}{\pi} > \dfrac{\pi}{6}$ $\quad \Rightarrow -\infty < \dfrac{n}{\pi} < \cot\dfrac{\pi}{6}$

$n < \pi\sqrt{3}$ $\qquad n \in N$, $\quad n_{max} = 5$

18.



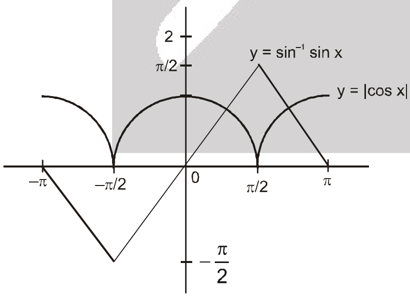
$\sec\tan^{-1}\left(\dfrac{2\sqrt{1-x^2}}{2x}\right) = \sec\tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right) = \dfrac{1}{x}\sum_{r=2}^{10} f\left(\dfrac{1}{r}\right)$ = 2 + 3 + ... + 10 = 54

19. $\sin^{-1}\left(\dfrac{3\sin 2\theta}{5+4\cos 2\theta}\right) = \dfrac{\pi}{2}$

Taking sin on both side $\dfrac{3\sin 2\theta}{5+4\cos 2\theta} = 1 \Rightarrow 3\sin 2\theta = 5 + 4\cos 2\theta \Rightarrow \dfrac{6\tan\theta}{1+\tan^2\theta} = 5 + 4\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$\tan^2\theta - 6\tan\theta + 9 = 0 \quad \Rightarrow \quad \tan\theta = 3$

20. Given equation is $|\cos x| = \sin^{-1}(\sin x)$ $\qquad -\pi \le x \le \pi$



Number of solution = 2

21. $\sin^{-1}x + \cos^{-1}(1 - x) = \sin^{-1}(-x)$

$2\sin^{-1}x + \cos^{-1}(1 - x) = 0$ $\qquad$ here $\quad x \in [0, 1]$

for $\quad x \in [0, 1]$ $2\sin^{-1}$ $\quad x \in [0, \pi]$ and $\cos^{-1}(1-x) \in \left[0, \dfrac{\pi}{2}\right]$

There sum is equal to zero when both terms equal to zero it gives x = 0 is only solution.

22. $\sqrt{\sum_{r=1}^{k} r^3} = \frac{k(k+1)}{2}$ $\Rightarrow$ $\cot^{-1}\left(1+2\sqrt{\sum_{k=1}^{k} r^3}\right) = \cot^{-1}\frac{1}{1+k(k+1)} = \cot^{-1}(k+1) - \cot^{-1}k$

$$\sum_{n=1}^{\infty}\left\{\frac{1}{\pi}\sum_{k=1}^{\infty}\cot^{-1}\left(1+2\sqrt{\sum_{r=1}^{k} r^3}\right)\right\}^n = \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n = \frac{1/4}{1-\frac{1}{4}} = \frac{1}{3}$$

PART - III

1. Obviously

2. n R m $\Rightarrow$ n is factor of m

(i) R is reflexive (ii) R is not symmetric because $\frac{m}{n}$ = K but $\frac{n}{m}$ N (iii) R is transitive.

3. $[2-x] + 2[x-1] \geq 0$

$\Rightarrow$ $2 + [-x] + 2[x] - 2 \geq 0$

$2[x] + [-x] \geq 0$

case-I : $x \in I$ then $2I - I \geq 0$

$\Rightarrow$ $I \geq 0$

so $x \in \{0, 1, 2, 3 .....\}$ ... (1)

case-II :x I then $I - 1 \geq 0$

$\Rightarrow$ $I \geq 1$ $\Rightarrow$ $I = 1, 2, 3 .....$

so $x \in (1, 2) \cup (2, 3) \cup (3, 4)$ ...... (2)

by (1) & (2)

$x \in \{0\} \cup [1, \infty)$

4. $\left(\frac{1}{3}\right)^{\{x\}} > \left(\frac{1}{3}\right)^{1/2}$ $\Rightarrow$ $\{x\} < \frac{1}{2}$

$\therefore$ solutions are $x = \pi$, $x = 2 + \frac{1}{\sqrt[3]{9}}$, $x = \frac{e}{2}$

5. $[x] \leq 3$ $\Rightarrow$ $x \in (-\infty, 4)$

$[x] < 4$ $\Rightarrow$ $x \in (-\infty, 4)$

$[x] > 3$ $\Rightarrow$ $x \in (4, \infty)$

$[x] \geq 4$ $\Rightarrow$ $x \in (4, \infty)$

$[x] + [-x] \geq 0 \Rightarrow [x] + [-x] = 0$ $\Rightarrow$ $x \in z$

$\{x\} + \{-x\} \leq 0$ $\{x\} + \{-x\} = 0$ $\Rightarrow$ $x \in z$

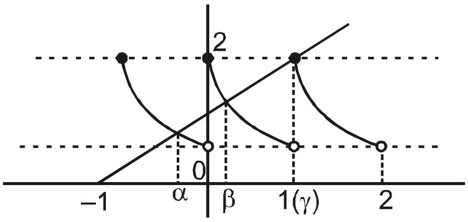
Hence (C)

$\text{sgn}(x^2 + 1) > 0$ $\Rightarrow$ $x \in R$

$x^2 + 7x + 43 > 0$ $\Rightarrow$ $x \in R$

Hence (D)

6. $2e^{-\{x\}} = x + 1$



$-1 < \alpha < 0 < \beta < \gamma = 1 \quad \Rightarrow \quad \alpha\beta\gamma < 0 \quad \& \quad \alpha + \beta + \gamma > 0.$

7. $\text{sgn}(x^2 - 6x + p) = q$

If $q = 0$, $s = 2$ $\quad \Rightarrow \quad$ $x^2 - 6x + p = 0$ has 2 distinct real roots.

If $q = 0$, $s = 0$ $\quad \Rightarrow \quad$ $x^2 - 6x + p = 0$ has imaginary roots.

$\text{sgn}(x^2 - 6x + p) = q$

8. $\Rightarrow \quad x \in I \quad \Rightarrow \quad [x] = x$

$\Rightarrow \quad |3x - 4x| = 4 \quad \Rightarrow \quad |x| = 4 \quad \Rightarrow \quad x = \pm 4$

Sum P = 0

Product Q = –16

9. $f(x) = \sin \log \left(\dfrac{\sqrt{4-x^2}}{1-x}\right)$

$4 - x^2 > 0$ or $x \in (-2, 2)$ and $\dfrac{\sqrt{4-x^2}}{1-x} > 0$

D : (–2, 1)

R : [–1, 1]

10. **(i)** $f(x) = \sqrt{x-1} + 2\sqrt{3-x}$

D : $x - 1 \geq 0 \quad \& \quad 3 - x \geq 0 \quad \Rightarrow \quad x \in [1, 3]$

Range: $f'(x) = \dfrac{1}{2\sqrt{x-1}} - \dfrac{1}{\sqrt{3-x}} = 0$ or $f'(x) = 0$ at $x = \dfrac{7}{5}$

$f'\left(\dfrac{7^-}{5}\right) > 0 \quad \& \quad f'\left(\dfrac{7^+}{5}\right) < 0 \quad \Rightarrow \quad$ maxima at $x = \dfrac{7}{5}$; Range : $\left[\sqrt{2}, \sqrt{10}\right]$

11. $[2 \cos x] + [\sin x] = -3$

$-2 \leq 2 \cos x \leq 2 \Rightarrow \quad [2 \cos x] = 2, 1, -1, -2$

$-1 \leq \sin x \leq 1 \quad \Rightarrow \quad [\sin x] = 1, 0, -1$

Equation holds true for $[2 \cos x] = -2$ and $[\sin x] = -1$

$\Rightarrow \quad -1 \leq \cos x < -\dfrac{1}{2} \quad$ and $\quad -1 \leq \sin x < 0$

$\Rightarrow \quad x \in \left(\dfrac{2\pi}{3}, \dfrac{4\pi}{3}\right) \quad$ and $\quad x \in (\pi, 2\pi) \quad \Rightarrow \quad x \in \left(\pi, \dfrac{4\pi}{3}\right)$

$f(x) = \sin x + \sqrt{3} \cos x$

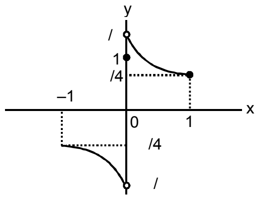
$f(x) = 2 \sin\left(x + \dfrac{\pi}{3}\right)$

$\dfrac{4\pi}{3} < x + \dfrac{\pi}{3} < \dfrac{5\pi}{3} \quad \Rightarrow \quad -1 \leq \sin\left(x + \dfrac{\pi}{3}\right) < -\dfrac{\sqrt{3}}{2} \quad \Rightarrow \quad -2 \leq 2\sin\left(x + \dfrac{\pi}{3}\right) < -\sqrt{3}$

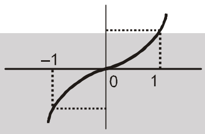
Hence range is $\left[-2, -\sqrt{3}\right]$

12. Domain D $\in$ [–1, 1]

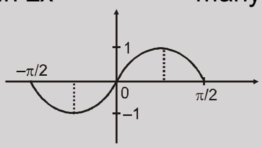
(A) $f(x) = \begin{cases} \tan^{-1}\dfrac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ many -one



(B) g (x) = x^3 one - one

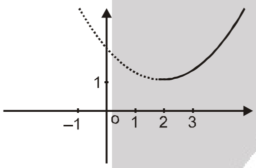


(C) h(x) = sin 2x many - one



(D) k(x) = sin $\left(\dfrac{\pi x}{2}\right)$ one-one function

13. f(x) = $(x^3 - x)$ Q(x) + $ax^2 + bx + c$
f(0) = 1 = c
f(1) = a + b + c = 3
f(–1) = a – b + c = –1 $\Rightarrow$ a = 0, b = 2, c = 1 $\Rightarrow$ g(x) = 2x + 1



14.
f : [2, ∞) $\to$ Y
f(x) = $x^2 - 4x + 5$
f(x) = $(x - 2)^2 + 1$
For given domain by graph range is [1, ∞)
For function to be onto codomain y = [1, ∞)

15. f(x) = $\begin{cases} x-1, \text{ x is even} \\ x+1, \text{ x is odd} \end{cases}$, which is clearly are one-one and onto.

16. (A) f(x) = $e^{1/2\, \ell n\, x} = \sqrt{x}$, D : x > 0
g(x) = $\sqrt{x}$, D : x $\geq$ 0

(B) tan (tan x) D : x $\neq \pm (2n+1)\dfrac{\pi}{2}$ & tan x $\neq \pm (2n+1)\dfrac{\pi}{2}$

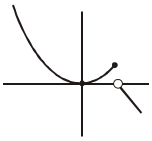
cot (cot x) D : x $\neq \pm n\pi$ & cot x $\neq \pm n\pi$

(C) f(x) = $\cos^2x + \sin^4x = \cos^2x + (1 - \cos^2x)^2 = 1 - \cos^2x + \cos^4x = \sin^2x + \cos^4x$
g(x) = $\sin^2x + \cos^4x$

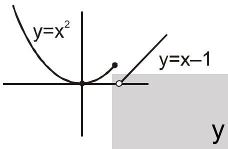
(D) f(x) = $\dfrac{|x|}{x}$, D : x $\neq$ 0 $\Rightarrow$ g(x) = sgn (x), D : x $\in$ R

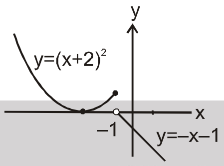
17. $f(x) = \dfrac{a^x - 1}{x^n(a^x + 1)}$ $\quad\because\quad$ $f(x) = f(-x)$

$\Rightarrow \quad \dfrac{1 - a^x}{(-x)^n(1 + a^x)} = \dfrac{a^x - 1}{x^n(a^x + 1)} \quad \Rightarrow \quad (-x)^n = -x^n \quad \Rightarrow \quad n = -\dfrac{1}{3}$



18. $y = f(x)$





$y = |f(x)|$

$y = |f(x)| + f(x + 2) = \begin{cases} x^2 + (x+2)^2 & x \le -1 \\ x^2 + (-x-1) & -1 < x \le 1 \\ x - 1 - x - 1 & x > 1 \end{cases}$; $\quad y = |f(x)| + f(x + 2) = \begin{cases} 2x^2 + 4x + 4 & x \le -1 \\ x^2 - x - 1 & -1 < x \le 1 \\ -2 & x > 1 \end{cases}$

19. $f(-x) = \begin{cases} 0 & x = 0 \\ -x^2 \sin\left(\frac{\pi}{x}\right) & x \in (-1, 1) - \{0\} \\ -x\,|x| & |x| > 1 \end{cases} = -f(x)$ odd function

20. $g(x) = x^3 + \tan x + \left[\dfrac{x^2+1}{P}\right] \quad \Rightarrow \quad g(-x) = (-x)^3 + \tan(-x) + \left[\dfrac{(-x)^2+1}{P}\right]$

$\Rightarrow \quad g(-x) = -x^3 - \tan x + \left[\dfrac{x^2+1}{P}\right] \Rightarrow \quad g(x) + g(-x) = 0$

because g(x) is a odd function

$\therefore \quad \left(-x^3 - \tan x + \left[\dfrac{x^2+1}{P}\right]\right) + \left(-x^3 - \tan x + \left[\dfrac{x^2+1}{P}\right]\right) = 0$

$\Rightarrow \quad 2\left[\dfrac{(x^2+1)}{P}\right] = 0 \Rightarrow 0 \le \dfrac{x^2+1}{P} < 1$

Now $x \in [-2, 2] \Rightarrow 0 < \dfrac{5}{P} < 1 \Rightarrow P > 5$

21. $f : R \to [-1, 1] \quad \Rightarrow \quad f(x) = \sin\left(\dfrac{\pi}{2}[x]\right) = \begin{cases} -1 , & -1 \le x < 0 \\ 0 , & 0 \le x < 1 \\ 1 , & 1 \le x \le 2 \end{cases}$

Many - one function $\qquad$ into function

Also $f(x + 4) = \sin\left(\dfrac{\pi}{2}[x+4]\right) = \sin\left(2\pi + \dfrac{\pi}{2}[x]\right) = \sin\left(\dfrac{\pi}{2}[x]\right) = f(x)$ and hence periodic

22. $f(x) = \dfrac{2x(\sin x + \tan x)}{2\left[\dfrac{x}{\pi}\right] + 1}$

if $x = n\pi$, $f(n\pi) = 0$ $\quad$ if $x \ne n\pi$ $\qquad \left[\dfrac{-x}{\pi}\right] = -\left(1 + \left[\dfrac{x}{\pi}\right]\right)$

$\therefore \quad f(-x) = -f(x)$ odd function

$f(0) = f(\pi)$ hence many one

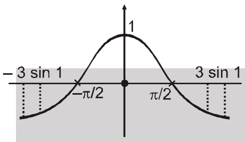
23. **(A)** $f(x) = \sin x + \cos x$

$f(-x) = \cos x - \sin x$ nither odd nor even

(B) $f(x + \pi) = \cos(\sin(x + \pi)) + \cos(\cos(x + \pi)) \Rightarrow f(x + \pi) = \cos(\sin x) + \cos(\cos x) = f(x)$

$$f\left(x+\frac{\pi}{2}\right) = \cos\left(\sin\left(x+\frac{\pi}{2}\right)\right) + \cos\left(\cos\left(x+\frac{\pi}{2}\right)\right) = \cos(\cos x) + \cos(\sin x) = f(x)$$

fundamental period = $\frac{\pi}{2}$

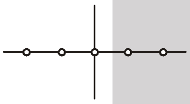


(C) $f(x) = \cos(3 \sin x)$, $\quad x \in [-1, 1]$

$-3 \le 3 \sin x \le 3 \Rightarrow \quad \cos(3) \le \cos(3 \sin x) \le 1$

$\therefore$ Range is $[\cos(3), 1]$ $[\cos(3), 1]$

24. $f(x) = \frac{\sin(\pi[x])}{\{x\}} = 0, \ x \notin I$



(A) By graph fundamental period is one

(B) $f(-x) = 0 = f(x)$ $\quad \therefore$ even function

(C) Range $y \in \{0\}$

(D) $y = \text{sgn} \ - 1, \ x \notin I$

$y = \text{sgn}\left(\text{sgn}\frac{\{x\}}{\sqrt{\{x\}}}\right)(1) - 1 \quad \Rightarrow \quad y = 1 - 1$

$y = 0, \ x \notin I$ $\quad$ Identical to $f(x)$

25. Clearly $h(x) = 2a$
so neither one-one nor onto

26. **(A)** $f(x) = e^{\ell n(\sec^{-1} x)} = \sec^{-1}x, \quad x \in (-\infty, -1] \cup (1, \infty)$
$g(x) = \sec^{-1}x, \quad x \in (-\infty, -1] \cup [1, \infty)$ non-identical functions

(B) $f(x) = \tan(\tan^{-1} x) = x, x \in R$
$g(x) = \cot(\cot^{-1} x) = x, x \in R$ identical functions

(C) $f(x) = \text{sgn}(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases} \Rightarrow g(x) = \text{sgn}(\text{sgn}\, x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$ Identical functions

(D) $f(x) = \cot^2 x \cdot \cos^2 x, \quad x \in R - \{n\pi\}, \quad n \in I$
$g(x) = \cot^2 x - \cos^2 x \quad = \cot^2 x(1 - \sin^2 x) = \cot^2 x \cdot \cos^2 x$
$x \in R - \{n\pi\}, \quad n \in I$ Identical functions

27. $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \Rightarrow x = y = z = 1 \Rightarrow x^{100} + y^{100} + z^{100} - \frac{9}{x^{101}+y^{101}+z^{101}} = 0.$

28. $x = \text{cosec } \tan^{-1} \cos \cot^{-1} \sec \sin^{-1} a = \text{co sec } \tan^{-1} \cos \cot^{-1}\left(\frac{1}{\sqrt{1-a^2}}\right)$

$= \text{cosec }\left\{\tan^{-1}\left(\frac{1}{\sqrt{2-a^2}}\right)\right\} = \sqrt{3-a^2}$

again $y = \sec \cot^{-1} \sin \tan^{-1} \text{cosec } \cos^{-1} a \Rightarrow y = \sec \cot^{-1} \sin \tan^{-1}\left(\frac{1}{\sqrt{1-a^2}}\right)$

$= \sec \cot^{-1}\left(\frac{1}{\sqrt{2-a^2}}\right) = \sqrt{(3-a^2)} \Rightarrow x = y = \sqrt{3-a^2}$

29. $x^2 - x - 2 > 0$ α satisfies it $\Rightarrow$ $\alpha^2 - \alpha - 2 > 0$ $\Rightarrow (\alpha - 2)(\alpha + 1) > 0$ $\Rightarrow$ $\alpha < -1$ or $\alpha > 2$
so C & D are correct.

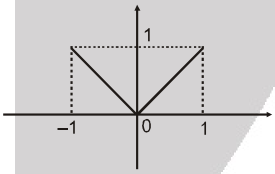
30. $f(x) = \ell n\,(\sin^{-1}(\log_2 x))$ Domain $0 < \log_2 x \le 1$, $x \in (1, 2]$ Range $\left(-\infty, \ell n\frac{\pi}{2}\right]$

31. $f : [-1, 1] \to [-1, 1]$

(A) $f(x) = \sin(\sin^{-1}x) = x$, $x \in [-1, 1]$ Bijective function $y \in [-1, 1]$

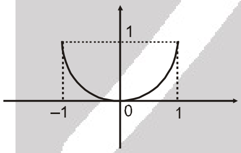
(B) $f(x) = \frac{2}{\pi}\sin^{-1}(\sin x) = \frac{2}{\pi}$, $x \in [-1, 1]$ Not bijective $y \in \left[\frac{-2}{\pi}, \frac{2}{\pi}\right]$

(C) $f(x)\,(\text{sgn } x)\,(\ell n\, e^x) = \begin{cases} x & , \; x \in (0,1] \\ 0 & , \; x = 0 \\ -x & , \; x \in [-1,0) \end{cases}$



Not bijective

(D)



$f(x) = x^3 \text{ sgn } x = \begin{cases} x^3 & , \; x > 0 \\ 0 & , \; x = 0 \\ -x^3 & , \; x < 0 \end{cases}$

Not bijective

32. $\sin \cot^{-1} \cos \tan^{-1} t = \sin \cot^{-1} \frac{1}{\sqrt{1+t^2}} = \frac{\sqrt{1+t^2}}{\sqrt{2+t^2}}$

similarly $\cos \tan^{-1} \sin \cot^{-1} \sqrt{2}\, t = \frac{\sqrt{1+2t^2}}{\sqrt{2+2t^2}}$

so $\frac{1}{\sqrt{2}}\left\{\frac{\sin\cot^{-1}\cos\tan^{-1}t}{\cos\tan^{-1}\sin\cot^{-1}\sqrt{2}\,t}\right\}.\left\{\sqrt{\frac{1+2t^2}{2+t^2}}\right\} = \frac{\frac{1}{\sqrt{2}}\frac{\sqrt{1+t^2}}{\sqrt{2+t^2}} \times \frac{\sqrt{1+2t^2}}{\sqrt{2+t^2}}}{\frac{\sqrt{1+2t^2}}{\sqrt{2}\sqrt{1+t^2}}} = \frac{1+t^2}{2+t^2} = 1 - \frac{1}{2-t^2}$

$0 < \frac{1}{t^2+2} \le \frac{1}{2}$; $\frac{1}{2} \le 1 - \frac{1}{t^2+2} < 1$

33. $\tan^{-1}\frac{\sqrt{1-x^2}}{1+x}$ since $0 < x < 1 = \tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$ (let $\cos^{-1} x = \theta$ $0 < \theta < \frac{\pi}{2}$)

$= \tan^{-1} \tan\frac{\theta}{2}$ $\quad \because \quad \frac{\theta}{2} \in \left(0, \frac{\pi}{4}\right) = \frac{\theta}{2} = \frac{1}{2}\cos^{-1} x$ ....(1)

also $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$\cos\frac{\theta}{2} = \sqrt{\frac{1+x}{2}}$ (taking $\cos^{-1}$ on both side)

$\cos^{-1}\cos = \frac{\theta}{2} = \cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$ since $\frac{\theta}{2} \in \left(0, \frac{\pi}{4}\right)$

$\Rightarrow \frac{\theta}{2} = \cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$ ...(2)

similarly $\sin\frac{\theta}{2} = \sqrt{\frac{1-x}{2}}$

$\sin^{-1}\sin\frac{\theta}{2} = \frac{\theta}{2} = \sin^{-1}\sqrt{\frac{1-x}{2}}$ ...(3)

also $\frac{\theta}{2} = \tan^{-1}\sqrt{\frac{1+x}{1-x}}$ ...(4)

34. Let $\theta = \cos^{-1}x$

$$f(x) = \theta + \cos^{-1}\left(\cos\left(\theta - \frac{\pi}{3}\right)\right) = \begin{cases} \theta + \theta - \frac{\pi}{3} & \frac{\pi}{3} < \theta \le \pi \\ \theta - \theta + \frac{\pi}{3} & 0 \le \theta \le \frac{\pi}{3} \end{cases} = \begin{cases} 2\cos^{-1}x - \frac{\pi}{3} & -1 \le x < \frac{1}{2} \\ \frac{\pi}{3} & \frac{1}{2} \le x \le 1 \end{cases}$$

(i) $f(2/3) = \frac{\pi}{3}$

(ii) $f(1/3) = 2\cos^{-1}\frac{1}{3} - \frac{\pi}{3}$

35. $\sum_{n=1}^{\infty} \tan^{-1}\frac{4n}{n^4 - 2n^2 + 2} = \lim_{k\to\infty}\sum_{n=1}^{k}\left\{\tan^{-1}(n+1)^2 - \tan^{-1}(n-1)^2\right\}$

$= \lim_{k\to\infty}\left\{\tan^{-1}(k+1)^2 + \tan^{-1}k^2 - \tan^{-1}1 - \tan^{-1}0\right\} = \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{4} - 0 = \frac{3\pi}{4}$

Also $\tan^{-1}2 + \tan^{-1}3 = \pi + \tan^{-1}\left(\frac{3+2}{1-3.2}\right)$.

Since $xy = 6 > 1 = \frac{3\pi}{4}$ and $\sec^{-1}\left(-\sqrt{2}\right) = \frac{3\pi}{4}$

36. $\tan x = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$

If $\tan \alpha = \frac{1}{\sqrt{2}} \Rightarrow x = n\pi \pm \alpha \qquad n \in z$

37. If $-1 \le x < 0$, then $-\frac{\pi}{2} \le \sin^{-1}x < 0$. Also $0 < 2\cot^{-1}(y^2 - 2y) < 2\pi$

$\therefore \quad -\frac{\pi}{2} < \sin^{-1} x + 2\cot^{-1}(y^2 - 2y) < 2\pi$

$\therefore$ there is no solution in this case.

thus x can not be negative ....(i)

Now if $x \ge 0$, then $0 \le \sin^{-1}x \le \frac{\pi}{2}$

$\Rightarrow \quad \frac{3\pi}{4} \le \cot^{-1}(y^2 - 2y) < \pi$

$\Rightarrow \quad y^2 - 2y \le -1$

$\Rightarrow \quad y = 1$

since for y = 1, we have $2\cot^{-1}(y^2 - 2y) = 2\cot^{-1}(-1) = \frac{3\pi}{2}$

$\therefore \quad \sin^{-1} x = \frac{\pi}{2}$ i.e. $x = 1$

$\therefore$ the solution is x = 1, y = 1

PART - IV

1. Number of one-one functions = 0

2. Total number of function = $3^5 = 243$

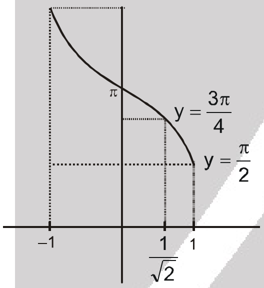
Number of into function = $3 + 3(2^5 - 2) = 93$

total number of onto function = 243 – 93 = 150

3. $\Rightarrow \quad g(6) \le g(7) \le g(8)$

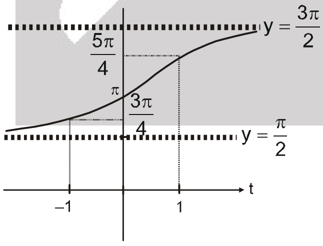
g(6)	g(7)	g(8)	No.of maps
1	1	1,2,3,4,5	5
	2	2,3,4,5	4
	3	3,4,5	3
	4	4,5	2
	5	5	1
2	2	2,3,4,5	4
	3	3,4,5	3
	4	4,5	2
	5	5	1
3	3	3,4,5	3
	4	4,5	2
	5	5	1
4	4	4,5	2
	5	5	1
5	5	5	1

4.



5. Clearly $\pi - x$.

6. $= \tan^{-1}(t)$, $t = -x \in [-1, 1]$



EXERCISE # 3

PART - I

1. $f(x) = x^2$; $g(x) = \sin x$ $\Rightarrow$ $gof(x) = \sin x^2$ $\Rightarrow$ $gogof(x) = \sin(\sin x^2)$

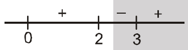
$\Rightarrow$ $(fogogof)(x) = (\sin(\sin x^2))^2 = \sin^2(\sin x^2)$

Now $\sin^2(\sin x^2) = \sin(\sin x^2)$ $\Rightarrow$ $\sin(\sin x^2) = 0, 1$

$\Rightarrow$ $\sin x^2 = n\pi, (4n+1)\frac{\pi}{2}$; $\eta \in I$ $\Rightarrow$ $\sin x^2 = 0$

$\Rightarrow$ $x^2 = n\pi$ $\Rightarrow$ $x = \pm\sqrt{n\pi}$; $n \in W$

2. $\tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{\cos 2\theta}}\right) = \sin^{-1}\left(\dfrac{\sin\theta}{\cos\theta}\right)$ $\therefore$ $f(\theta) = \tan\theta$ $\therefore$ $\dfrac{df}{d\tan\theta} = 1$

3. 

F : [0, 3] $\rightarrow$ [1, 29]

$f(x) = 2x^3 - 15x^2 + 36x + 1$

$f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x-2)(x-3)$

in given domain function has local maxima, it is many-one

Now at $x = 0$ $f(0) = 1$

$x = 2$ $f(2) = 16 - 60 + 72 + 1 = 29$

$x = 3$ $f(3) = 54 - 135 + 108 + 1 = 163 - 135 = 28$

Has range = [1, 29]

Hence given function is onto

4*. $\cos 4\theta = \dfrac{1}{3}$ $\Rightarrow 2\cos^2 2\theta - 1 = \dfrac{1}{3} \Rightarrow \cos^2 2\theta\ \dfrac{2}{3} =$ $\Rightarrow \cos 2\theta = \pm\sqrt{\dfrac{2}{3}}$

Now $f(\cos 4\theta) = \dfrac{2}{2-\sec^2\theta} = \dfrac{1+\cos 2\theta}{\cos 2\theta} = 1 + \dfrac{1}{\cos 2\theta} \Rightarrow f\left(\dfrac{1}{3}\right) = 1 \pm \sqrt{\dfrac{3}{2}}$

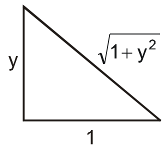
5. $\cot \displaystyle\sum_{n=1}^{23} \cot^{-1}(1+2+4+6+......+2n)$ $\Rightarrow$ $\cot \sum \cot^{-1}(1 + n(n+1))$

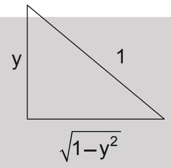
$\cot \sum \tan^{-1} \dfrac{(n+1)-n}{1+n(n+1)}$ $\Rightarrow$ $\cot \displaystyle\sum_{n=1}^{23}\left(\tan^{-1}(n+1) - \tan^{-1} n\right)$

$\cot(\tan^{-1}24 - \tan^{-1}1)$ $\Rightarrow$ $\cot\left(\tan^{-1}\dfrac{24-1}{1+24}\right)$ $\Rightarrow$ $\cot\left(\cot^{-1}\dfrac{25}{23}\right) = \dfrac{25}{23}$

6. **(P)** $$\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1}y)+y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y)+\tan(\sin^{-1}y)}\right)^2+y^4\right)^{1/2}$$

$$=\left[\frac{1}{y^2}\left[\frac{\left(\frac{1}{\sqrt{1+y^2}}+\frac{y.y}{\sqrt{1+y^2}}\right)}{\left(\frac{\sqrt{1-y^2}}{y}+\frac{y}{\sqrt{1-y^2}}\right)}\right]^2+y^4\right]^{1/2}$$





$$=\left(\frac{1}{y^2}.y^2(1-y^4)+y^4\right)^{1/2}=1$$

Ans. 4

(Q) cos x + cos y = – cos z

sin x + sin y = – sin z   square and add

2 + 2 cos (x – y) = 1

$\Rightarrow$ cos (x – y) = –1/2

$\Rightarrow$ $2\cos^2\left(\frac{x-y}{2}\right)-1=-1/2$, $\Rightarrow \cos\left(\frac{x-y}{2}\right)=1/2$

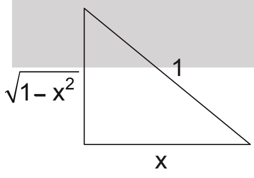
(R) $\cos 2x\left(\cos\left(\frac{\pi}{4}-x\right)-\cos\left(\frac{\pi}{4}+x\right)\right)+2\sin^2 x=2\sin x\cos x$

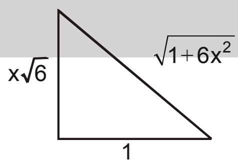
cos 2x (sin x) + 2 sin² x = 2 sin x cos x $\Rightarrow$ $\sqrt{2}$ sin x [cos 2x + $\sqrt{2}$ sin x – $\sqrt{2}$ cos x] = 0

Either sin x = 0 OR $\cos^2 x-\sin^2 x=(\cos x-\sin x)$

$\boxed{\sec x=1}$ OR cos x = sin x $\Rightarrow$ $\boxed{\sec x=\sqrt{2}}$

(S) cot (sin⁻¹) = sin (tan⁻¹(x $\sqrt{6}$))





$\frac{x}{\sqrt{1-x^2}}=\frac{x\sqrt{6}}{\sqrt{1+6x^2}}$ $\Rightarrow$ $1+6x^2=6-6x^2 \Rightarrow 12x^2=5$

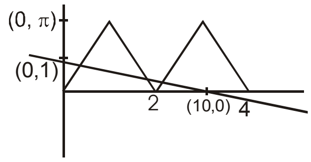
$x=\sqrt{\frac{5}{12}}=\frac{1}{2}\sqrt{\frac{5}{3}}$

7. **(i)** f(–x) = –f(x) so it is odd function

(ii) $f'(x)=3(\log(\sec x+\tan x))^2\,\frac{1}{(\sec x+\tan x)}(\sec x\tan x+\sec^2 x)>0$

(iii) Range of f(x) is R as $f\left(-\frac{\pi}{2}\right)\Rightarrow-\infty\Rightarrow\ f\left(\frac{\pi}{2}\right)\Rightarrow\infty$

8. $f(x) = (\sin^{-1})\ x \in [0, 4\pi]\ \&\ f(x) = \dfrac{10-x}{10} = 1 - \dfrac{x}{10}$



so, 3 solution.

9. $\alpha = 3\sin^{-1}\dfrac{6}{11} > 3\sin^{-1}\dfrac{6}{12}$ and $\beta = 3\cos^{-1}\dfrac{4}{9} > 3\cos^{-1}\dfrac{4}{8}$

$\Rightarrow \quad \alpha > \dfrac{\pi}{2} \quad \& \quad \beta > \pi \quad \Rightarrow \quad \alpha + \beta > \dfrac{3\pi}{2}$

10. $\sin^{-1}\left(\sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty}\left(\dfrac{x}{2}\right)^i\right) = \dfrac{\pi}{2} - \cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\dfrac{x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i\right)$

$\left(\dfrac{x^2}{1-x} - x\dfrac{\frac{x}{2}}{1-\frac{x}{2}}\right) = \dfrac{x}{1+x} + \dfrac{\left(-\frac{x}{2}\right)}{1+\frac{x}{2}} \quad \dfrac{x^2}{1-x} - \dfrac{x^2}{2-x} = \dfrac{x}{1+x} - \dfrac{x}{2+x}$

$\dfrac{x^2}{1-x} - \dfrac{x}{1+x} = \dfrac{x^2}{2-x} - \dfrac{x}{2+x}$

$\dfrac{x(1+x)-(1-x)}{1-x^2} = \dfrac{2x+x^2-2+x}{4-x^2}$ or $x = 0$

$\dfrac{x^2+2x-1}{1-x^2} = \dfrac{x^2+3x-2}{4-x^2} \Rightarrow x^3 + 2x^2 + 5x - 2 = 0$

Let $f(x) = x^3 + 2x^2 + 5x - 2$

$f'(x) > 0$

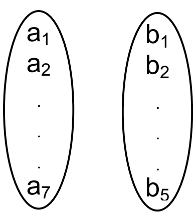
$f(0) = -2$ and $f(1/2) = 9/8$ so one root in $\left(0, \dfrac{1}{2}\right) \Rightarrow 2$ roots

11. $n(X) = 5$

$n(Y) = 7$

$\alpha \to$ Number of one-one function X to Y $= {}^7C_5 \times 5! = 21 \times 120 = 2520$

$\beta \to$ Number of onto function Y to X



1, 1, 1, 1, 3   1, 1, 1, 2, 2

$\dfrac{7!}{3!\,4!} \times 5! + \dfrac{7!}{(2!)^3 3!} \times 5! = ({}^7C_3 + 3\ .\ {}^7C_3)\ 5! = 4 \times {}^7C_3 \times 5!$

$\dfrac{\beta - \alpha}{5!} = 4 \times {}^7C_3 - {}^7C_5 = 4 \times 35 - 21 = 119$

12. $E_1 : \dfrac{x}{x-1} > 0 \quad \Rightarrow \quad x \in (-\infty, 0) \cup (1, \infty)$

$E_2 : -1 \le \ell n\left(\dfrac{x}{x-1}\right) \le 1 \quad \Rightarrow \quad \dfrac{1}{e} \le \dfrac{x}{x-1} \le e \Rightarrow \dfrac{1}{e} \le 1 + \dfrac{1}{x-1} \le e$

$\dfrac{1}{e} - 1 \le \dfrac{1}{x-1} \le e - 1 \quad \Rightarrow \quad (x-1) \in \left(-\infty, \dfrac{e}{1-e}\right] \cup \left[\dfrac{1}{e-1}, \infty\right)$

$x \in \left(-\infty, \dfrac{1}{e-1}\right] \cup \left[\dfrac{e}{e-1}, \infty\right)$

Now $\dfrac{x}{x-1} \in (0, \infty) - \{1\} \ \forall\ x \in E_1 \quad \Rightarrow \quad \ell n\left(\dfrac{x}{x-1}\right) \in (-\infty, \infty) - \{0\}$

$\sin^{-1}\left(\ell n\left(\dfrac{x}{x-1}\right)\right) \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$

13. Evaluate $\sec^{-1}\left[\dfrac{1}{4}\sum_{k=0}^{10}\sec\left(\dfrac{7\pi}{12} + \dfrac{k\pi}{12}\right)\sec\left(\dfrac{7\pi}{12} + (k+1)\dfrac{\pi}{2}\right)\right]$

Given exp $= \sec^{-1}\left[-\dfrac{1}{4}\sum_{k=0}^{10}\sec\left(\dfrac{7\pi}{12} + \dfrac{k\pi}{12}\right)\operatorname{cosec}\left(\dfrac{7\pi}{12} + \dfrac{k\pi}{2}\right)\right]$

$= \sec^{-1}\left[-\dfrac{1}{4}\sum_{k=0}^{10}\dfrac{2}{\sin\left(\dfrac{7\pi}{6} + k\pi\right)}\right] = \sec^{-1}\left[-\dfrac{1}{2}\sum_{k=0}^{10}\dfrac{1}{(-1)^{k+1}\sin\dfrac{\pi}{6}}\right] = \sec^{-1}\left(-\sum_{k=0}^{10}\dfrac{1}{(-1)^{k+1}}\right) = \sec^{-1}(1) = 0$

PART - II

1. $(x, x) \in R$ for $w = 1$

$\therefore$ R is reflexive

If $x \neq 0$, then $(0, x) \in R$ for $w = 0$ but $(x, 0) \notin R$ for any w

$\therefore$ R is not symmetric $\Rightarrow$ R is not equivalence relation

$\left(\dfrac{m}{n}, \dfrac{p}{q}\right) \in S \Rightarrow qm = pn \Rightarrow \dfrac{m}{n} = \dfrac{p}{q}$

(i) $\dfrac{m}{n} = \dfrac{m}{n} \Rightarrow \left(\dfrac{m}{n}, \dfrac{m}{n}\right) \in S \Rightarrow$ Reflexive

(ii) $\dfrac{m}{n} = \dfrac{p}{q} \Rightarrow \dfrac{p}{q} = \dfrac{m}{n} \Rightarrow$ symmetric

(iii) $\dfrac{m}{n} = \dfrac{p}{q}$ and $\dfrac{p}{q} = \dfrac{x}{y} \Rightarrow \dfrac{m}{n} = \dfrac{x}{y} \Rightarrow$ transitive $\Rightarrow$ S is equivalence relation

2. **Statement - 1 :**

(i) $x - x$ is an integer $\forall\ x \in R$ so A is reflexive relation.

(ii) $y - x \in I \Rightarrow x - y \in I$ so A is symmetric relation.

(iii) $y - x \in I$ and $z - y \in I \Rightarrow y - x + z - y \in I$

$\Rightarrow z - x \in I$ so A is transitive relation.

Therefore A is equivalence relation.

Statement - 2 :

(i) $x = \alpha x$ when $\alpha = 1 \Rightarrow$ B is reflexive relation

(ii) for $x = 0$ and $y = 2$, we have $0 = \alpha(2)$ for $\alpha = 0$

But $2 = \alpha(0)$ for no α

so B is not symmetric so not equivalence.

3. **for reflexive :** $(A, A) \in R \Rightarrow A = P^{-1} A P$
which is true for $P = I$

$\therefore$ reflexive

for symmetry: As $(A, B) \in R$ for matrix P

$A = P^{-1} BP \Rightarrow PA = PP^{-1} BP \Rightarrow PAP^{-1} = IBPP^{-1}$

$\Rightarrow PAP^{-1} = IBI \Rightarrow PAP^{-1} = B \Rightarrow B = PAP^{-1}$

$\therefore$ $(B, A) \in R$ for matrix P^{-1}

$\therefore$ R is symmetric

for transitivity

$A = P^{-1} BP$ and $B = P^{-1}CP \Rightarrow A = P^{-1} (P^{-1} CP)P$

$\Rightarrow A = (P^{-1})^2 CP^2$

$\Rightarrow A = (P^2)^{-1} C(P^2)$

$\therefore$ $(A, C) \in R$ for matrix P^2

$\therefore$ R is transitive

so R is equivalence

4. $f(x) = \dfrac{1}{\sqrt{|x| - x}} \Rightarrow |x| - x > 0 \Rightarrow |x| > x \Rightarrow x < 0$

$\therefore$ $x \in (-\infty, 0)$ **Ans.**

5. $f(x) = (x - 1)^2 + 1, x \geq 1$

$f : [1, \infty) \to [1, \infty)$ is a bijective function $\Rightarrow y = (x - 1)^2 + 1 \Rightarrow (x - 1)^2 = y - 1 \Rightarrow x = 1 \pm \sqrt{y-1} \Rightarrow f^{-1}(y)$ $= 1 \pm \sqrt{y-1} \Rightarrow f^{-1}(x) = 1 + \sqrt{x-1}$ $\{\therefore x \geq 1\}$ so statement-2 is correct

Now $f(x) = f^{-1}(x) \Rightarrow f(x) = x \Rightarrow (x - 1)^2 + 1 = x \Rightarrow x^2 - 3x + 2 = 0 \Rightarrow x = 1, 2$ so statement-1 is correct

6. $2y = x + z \Rightarrow 2 \tan^{-1} y = \tan^{-1} x + \tan^{-1} (z)$ $\tan^{-1}\left(\dfrac{2y}{1-y^2}\right) = \tan^{-1}\left(\dfrac{x+z}{1-xz}\right)$

$\Rightarrow \dfrac{x+z}{1-y^2} = \dfrac{x+z}{1-xz} \Rightarrow y^2 = xz$ or $x + z = 0 \Rightarrow x = y = z$

7. If f(x) & g(x) are inverse of each other then, $g'(f(x)) = \dfrac{1}{f'(x)}$; $g'(f(x)) = 1 + x^5$

Here $x = g(y) \Rightarrow g'(y) = 1 + [g(y)]^5 \Rightarrow g'(x) = 1 + \{g(x)\}^5$

8. $\dfrac{-1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}}$ $x = \tan\theta$

$\dfrac{-\pi}{6} < \theta < \dfrac{\pi}{6}$ $\tan^{-1}y = \theta + \tan^{-1} \tan 2\theta = \theta + 2\theta = 3\theta$

$y = \tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ $y = \dfrac{3x - x^3}{1 - 3x^2}$.

9. $f(x) + 2f\left(\frac{1}{x}\right) = 3x$

S : f(x) = f(–x)

$f(x) + 2f\left(\frac{1}{x}\right) = 3x$ .....(1)

$x \to f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x}$ .....(2)

$(1) - 2 \times (2)\ -3f(x) = 3x - \frac{6}{x}\ f(x) = \frac{2}{x} - x$

Now $f(x) = f(-x)$

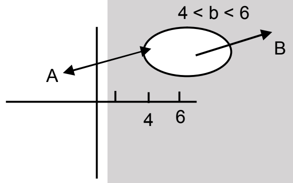
$\therefore \quad \frac{2}{x} - x = \frac{2}{x} + x\ \ \frac{4}{x} = 2x$

$\frac{2}{x} = x \quad \Rightarrow x = \pm\sqrt{2}$

Exactly two elements

10. $-1 < a - 5 > 1$

$4 < a < 6$



$A \subset B$

$\frac{(a-6)^2}{3^2} + \frac{(b-5)^2}{2^2} = 1$ It passes through (4, 6) $\Rightarrow \frac{16+9-36}{36} = \frac{25-36}{36} < 0$

11. $\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) = \frac{\pi}{2}\ \left(x > \frac{3}{4}\right)$

$\cos^{-1}\left(\frac{2}{3x} \times \frac{3}{4x} - \sqrt{1-\frac{4}{9x^2}}\sqrt{1-\frac{9}{16x^2}}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \frac{1}{2x^2} = \frac{\sqrt{9x^2-4}\sqrt{16x^2-9}}{12x^2} \Rightarrow \quad 6 = \sqrt{9x^2-4}\sqrt{16x^2-9}$

square both side $36 = 144x^4 - 81x^2 - 64x^2 + 36$

$\Rightarrow \quad 144x^4 = 145x^2$

$\Rightarrow \quad x^4 = \frac{145x^2}{144} \quad \Rightarrow \quad x = \pm\frac{\sqrt{145}}{12},\ \ 0$

$\therefore \quad x > \frac{3}{4}$ hence $x = \frac{\sqrt{145}}{12}$

12. $f_2(Jo(f_1(x))) = f_3(x)$

$\Rightarrow \quad 1 - J(f_1(x)) = f_3(x)$

$\Rightarrow \quad 1 - J\left(\frac{1}{x}\right) = \frac{1}{1-x}$

$\Rightarrow \quad J\left(\frac{1}{x}\right) = 1 - \frac{1}{1-x} = -\frac{x}{1-x}$

$\Rightarrow \quad J(x) = \frac{\frac{-1}{x}}{1-\frac{1}{x}} = \frac{1}{1-x}$

$\Rightarrow \quad J(x) = f_3(x)$

13. $\cot\sum_{n=1}^{19}\tan^{-1}\left(\frac{n+1-n}{1+n(n+1)}\right) = \cot(\tan^{-1}20 - \tan^{-1}1) = \frac{1}{\tan(\tan^{-1}20-\tan^{-1}1)} = \frac{1}{\frac{20-1}{1+(20).1}} = \frac{21}{19}$

Aliter : $\cot\sum_{n=1}^{19}\left(\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right) = \cot\sum_{n=1}^{19}\cot^{-1}(1+2)(2+3+4....n)]$

$= \cot\sum_{n=1}^{19}(\cot^{-1}(1+n(n+1)) = \cot\sum_{n=1}^{19}(\cot^{-1}(n^2+n+1)$

14. k = {4, 8, 12, 16, 20}

f(k) can take values from set {3, 6, 9, 12,15,18} this can be done is ${}^6C_5 \times 5!$ ways. = 6! ways

and options for remaining 15 elements of A = 15!

15. A = $\left\{2^{(x+2)(x^2-5x+6)} = 1; x \in I\right\}$; B = $\{-3 < 2x - 1 < 9 ; x \in I\}$

A = {–2, 2, 3}; B = {0, 1, 2, 3, 4}

n(A × B) = 15

Number of subsets = 2^{15}

16. $f(x) = \ell n\left(\frac{1-x}{1+x}\right)$

$f\left(\frac{2x}{1+x^2}\right) = \ell n\left(\frac{1-\frac{2x}{1+x^2}}{1+\frac{2x}{1+x^2}}\right) = \ell n\left(\frac{1+x^2-2x}{1+x^2+2x}\right) = \ell n\left(\left(\frac{1-x}{1+x}\right)^2\right) = 2\ell n\left(\frac{1-x}{1+x}\right) = 2f(x)$

17. $\tan\alpha = \frac{4}{3}$

$\tan\beta = \frac{1}{3} \Rightarrow \tan(\alpha-\beta) = \frac{\frac{4}{3}-\frac{1}{3}}{1+\frac{4}{3}\times\frac{1}{3}} = \frac{9}{13} \Rightarrow \sin(\alpha-\beta) = \frac{9}{5\sqrt{10}}$

$\Rightarrow \alpha - \beta = \sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

18. $2y = \left(\cot^{-1}\left(\dfrac{\sqrt{3}\cos x + \sin x}{\cos x - \sqrt{3}\sin x}\right)\right)^2$

$2y = \left(\cot^{-1}\left(\dfrac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x}\right)\right)^2$

$2y = \left(\cot^{-1}\left(\tan\left(\dfrac{\pi}{3} + x\right)\right)\right)^2$

$2y = \left(\dfrac{\pi}{2} - \tan^{-1}\left(\tan\left(\dfrac{\pi}{3} + x\right)\right)\right)^2 = \left(\dfrac{\pi}{2} - \left(\dfrac{\pi}{3} + x\right)\right)^2$

$2y = \left(x - \dfrac{\pi}{6}\right)^2$

$2y = x^2 - \dfrac{\pi}{3}x + \dfrac{\pi^2}{36}$

$y' = x - \dfrac{\pi}{6}$

19. $f_1(x) = \left(\dfrac{a^x + a^{-x}}{2}\right) \Rightarrow f_2(x) = \left(\dfrac{a^x - a^{-x}}{2}\right)$ so $f_1(x + y) + f_1(x - y) = \dfrac{1}{2}\left[a^{x+y} + \dfrac{1}{a^{x+y}} + a^{x-y} + \dfrac{1}{a^{x-y}}\right]$

$= \dfrac{1}{2}\left[a^x.a^y + \dfrac{1}{a^x.a^y} + \dfrac{a^x}{a^y} + \dfrac{a^y}{a^x}\right] = \dfrac{1}{2}(a^x + a^{-x})(a^y + a^{-y}) = 2f_1(x)\, f_1(y)$

20. $y = \dfrac{8^{2x} - 8^{-2x}}{8^{2x} + 8^{-2x}}$ $\quad \dfrac{1+y}{1-y} = \dfrac{8^{2x}}{8^{-2x}}$ $\quad 8^{4x} = \dfrac{1+y}{1-y}$

$4x = \log_8\left(\dfrac{1+y}{1-y}\right)$ $\quad x = \dfrac{1}{4}\log_8\left(\dfrac{1+y}{1-y}\right)$ $\quad f^{-1}(x) = \dfrac{1}{4}\log_8\left(\dfrac{1+x}{1-x}\right)$

21. $f(x) \begin{cases} \dfrac{x}{x^2+1}; & x \in (1,2) \\ \dfrac{2x}{x^2+1}; & x \in [2,3) \end{cases}$

$\therefore$ f(x) is a decreasing function

$\therefore$ $\mathbf{y} \in \left(\dfrac{2}{5}, \dfrac{1}{2}\right) \cup \left(\dfrac{6}{10}, \dfrac{4}{5}\right]$

$\Rightarrow$ $\mathbf{y} \in \left(\dfrac{2}{5}, \dfrac{1}{2}\right) \cup \left(\dfrac{3}{5}, \dfrac{4}{5}\right]$

22. $f'(x) = \tan^{-1}(\sec x + \tan x) = \tan^{-1}\left(\dfrac{1 + \sin x}{\cos x}\right) = \tan^{-1}\left(\dfrac{1 - \cos\left(\dfrac{\pi}{2} + x\right)}{\sin\left(\dfrac{\pi}{2} + x\right)}\right) = \tan^{-1}\left(\dfrac{2\sin^2\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)}{2\sin\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)\cos\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)}\right)$

$= \tan^{-1}\left(\tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)\right) = \dfrac{\pi}{4} + \dfrac{x}{2}$ $\quad (f'(x))dx = \dfrac{\pi}{4} + \dfrac{x}{2}dx$ $\quad f(x) = \dfrac{\pi}{4}x + \dfrac{x^2}{4} + c$ $\quad f(0) = c = 0 \Rightarrow f(x) = \dfrac{\pi}{4}x + \dfrac{x^2}{4}$

So $f(1) = \dfrac{\pi + 1}{4}$

HIGH LEVEL PROBLEMS (HLP)

1. Case-1 When $x \in I$

$\{-x\} = 0 \Rightarrow x = 3x \Rightarrow x = 0 \in I$

Case-2 When $x \notin I$

$\{-x\} = 1 - \{x\}$

$[x] + 2 - 2\{x\} = 3[x] + 3\{x\}$

$\{x\} = \dfrac{2-2[x]}{5} \in [0, 1) \Rightarrow 0 \le 2 - 2[x] < 5 \Rightarrow -\dfrac{3}{2} < [x] \le 1 \Rightarrow [x] = -1, 0, 1$

for $[x] = -1$

$\{x\} = \dfrac{4}{5} \Rightarrow x = -\dfrac{1}{5} \notin I$

for $[x] = 0$

$\{x\} = \dfrac{2}{5} \Rightarrow x = \dfrac{2}{5} \notin I$

for $[x] = 1$ $\{x\} = 0$

$x = 1 \in I$, so regect

2. $2x + 3[x] - 4\{-x\} = 4$

$\Rightarrow 2x + 3[x] - 4(-x - [-x]) = 4 \Rightarrow 2x + 3[x] + 4x + 4[-x] = 4$

Case-I : If $x \in I \Rightarrow 2x + 3x + 4x - 4x = 4$

$\Rightarrow 5x = 4 \Rightarrow x = \dfrac{4}{5} \notin I$

Case-II : If $x \notin I \Rightarrow [-x] = -[x] - 1$

$\Rightarrow 2x + 3[x] + 4x - 4[x] - 4 = 4 \Rightarrow 6x - [x] = 8$

$\Rightarrow 6x = [x] + 8$ ....(i)

$\Rightarrow 6[x] + 6\{x\} = [x] + 8 \Rightarrow 6\{x\} = 8 - 5[x] \Rightarrow \{x\} = \dfrac{8-5[x]}{6}$

$\Rightarrow 0 \le \dfrac{8-5[x]}{6} < 1 \Rightarrow 0 \le 8 - 5[x] < 6$

$\Rightarrow -8 \le -5[x] < -2 \Rightarrow \dfrac{2}{5} < [x] < \dfrac{8}{5} \Rightarrow [x] = 1$

by (i) $6x = 9$ $x = \dfrac{3}{2}$

3. **Case-I** $x \le \dfrac{1}{2}$

$-2x + 1 = 3[x] + 2\{x\}$

$\Rightarrow x = \dfrac{1}{4}$

Case-II $x > \dfrac{1}{2}$

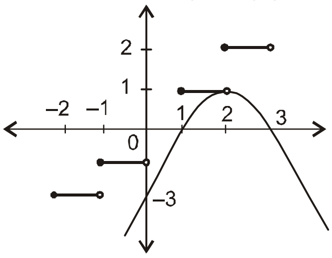
$2x - 1 = 3[x] + 2\{x\}$

$\Rightarrow 2[x] + 2\{x\} - 1 = 3[x] + 2\{x\}$

$[x] = -1$

$\Rightarrow x \in \phi$

4. $[x] = -x^2 + 4x - 3 = -(x-1)(x-3)$



No solution

5. Product of $\left[x - \frac{1}{2}\right]\left[x + \frac{1}{2}\right]$ is prime when one of them is 1 or –1 and other is prime or negative of prime

Case 1 :

So, $\left[x - \frac{1}{2}\right] = 1 \Rightarrow 1 \le x - \frac{1}{2} < 2 \Rightarrow \frac{3}{2} \le x < \frac{5}{2}$,

Now $2 \le x + \frac{1}{2} < 3 \quad \therefore \left[x + \frac{1}{2}\right] = 2$

Case 2 :

$\left[x + \frac{1}{2}\right] = -1 \Rightarrow -1 \le x + \frac{1}{2} < 0 \Rightarrow \frac{-3}{2} \le x < \frac{-1}{2}$,

Now $-2 \le x - \frac{1}{2} < -1 \quad \therefore \left[x - \frac{1}{2}\right] = -2$

Other cases will not give any result Solution is $\left[\frac{-3}{2}, \frac{-1}{2}\right) \cup \left[\frac{3}{2}, \frac{5}{2}\right)$

6. $[x]^2 + (x)^2 < 4$

if $x \in I$, then $[x] = (x) = x$

$\therefore$ $x^2 < 2$ i.e. $-\sqrt{2} < x < \sqrt{2}$ i.e. $x = -1, 0, 1$

if $x \notin I$, then $(x) = 1 + [x]$

$\therefore$ $[x]^2 + (1 + [x])^2 < 4$

i.e. $2[x]^2 + 2[x] - 3 < 0$

i.e. $\frac{-2-\sqrt{28}}{4} < [x] < \frac{-2+\sqrt{28}}{4}$ i.e. $\frac{-1-\sqrt{7}}{2} < [x] < \frac{-1+\sqrt{7}}{2}$

i.e. $[x] = -1, 0$

$\therefore$ Solution set is $[-1, 1]$ $\therefore$ $\lambda + \mu = 0$

7. $\left[\frac{x}{9}\right] = \left[\frac{x}{11}\right]$

99x 11x = 9n

$\frac{x}{9} - \left\{\frac{x}{9}\right\} = \frac{x}{11} - \left\{\frac{x}{11}\right\}$

$\frac{2x}{99} = \left\{\frac{x}{9}\right\} - \left\{\frac{x}{11}\right\} \in (-1, 1)$

$x \in \left(-\frac{99}{2}, \frac{99}{2}\right) \Rightarrow x \in \left(0, \frac{99}{2}\right)$

Here $x \in \{1, 2, 3, ... 8\} \cup \{11, 12, 13, 14, 15, ... 17\} \cup \{22, 23, ..., 26\} \cup \{33, 34, 35\} \cup \{44\}$

total positive integers = 24

8. $f(x) = \left[\frac{x}{15}\right]\left[\frac{-15}{x}\right]$

If $x \geq 15$, then $\left[-\frac{15}{x}\right] = -1$

$\therefore \quad f(x) = -\left[\frac{x}{15}\right]$ so $x \in [15, 90)$ $\frac{x}{15} \in [1, 6)$

$\therefore \quad f(x) = -1, -2, -3, -4, -5$ and if $x \in (0, 15)$, then $\left[\frac{x}{15}\right] = 0$

$\therefore \quad f(x) = 0$

Hence $f(x) \in \{-5, -4, -3, -2, -1, 0\}$

9. $\left[\frac{3}{x}\right] + \left[\frac{4}{x}\right] = 5$

$x \not\leq 0$

(i) $\left[\frac{3}{x}\right] = 5 \Rightarrow \frac{3}{x} \in [5, 6]$ $x \in \left[\frac{1}{2}, \frac{3}{5}\right]$ But $\left[\frac{4}{x}\right] \neq 0$

(ii) $\left[\frac{4}{x}\right] = 1 \Rightarrow \quad \frac{4}{x} \in [1, 2) \Rightarrow \quad x \in (2, 4] \Rightarrow \quad x = \phi$

$\left[\frac{3}{x}\right] = 4 \Rightarrow \quad \frac{3}{x} \in [4, 5) \Rightarrow \left(\frac{3}{5}, \frac{3}{4}\right]$

(iii) $\left[\frac{4}{x}\right] = 3 \Rightarrow \quad \frac{4}{x} \in [3, 4) \Rightarrow \frac{1}{x} \in \left[\frac{3}{4}, 1\right)$

$\Rightarrow \quad \frac{1}{x} \in \left[\frac{3}{4}, 1\right) \in x \in \left(1, \frac{4}{3}\right]$

$\left[\frac{3}{x}\right] = 2 \Rightarrow \quad \frac{3}{x} \in [2, 3) \Rightarrow \quad \frac{1}{x} \in \left[\frac{2}{3}, 1\right)$

$\Rightarrow \quad \frac{1}{x} \in \left[\frac{3}{4}, 1\right) \Rightarrow x \in \left(1, \frac{4}{3}\right]$ a = 1, b = 4, c = 3

10. $\frac{19}{6}, \frac{29}{12}, \frac{97}{24}$

Case -1 $x = \mathrm{I}$

$\frac{1}{\mathrm{I}} + \frac{1}{2\mathrm{I}} = \frac{1}{3}$

$\frac{3}{2\mathrm{I}} = \frac{1}{3}$ I = not possible

Case -2 $x = \mathrm{I} + f;\ f \in \left(0, \frac{1}{2}\right)$

$\frac{1}{\mathrm{I}} + \frac{1}{2\mathrm{I}} = f + \frac{1}{3}$

$\frac{3}{2\mathrm{I}} = f + \frac{1}{3}$

$\frac{9}{2(3f+1)} = \mathrm{I}$

$\because \quad f \in \left(0, \frac{1}{2}\right) \; 2(3f + 1) \in (2, 5)$

$\therefore \quad I \in \left(\frac{9}{5}, \frac{9}{2}\right) \; I = 2, 3, 4$

$I = 2 \; ; I = 3 \; ; I = 4$

$f = \frac{5}{12} \; ; f = \frac{1}{6} \; ; f = \frac{1}{24}$

$\therefore \quad x = \frac{29}{12} \; ; x = \frac{19}{6} \; ; x = \frac{97}{24}$

Case-3 $x = I + f \; ; f \in \left[\frac{1}{2}, 1\right) \; \frac{1}{I} + \frac{1}{2I+1} = f + \frac{1}{3} \; \frac{3I+1}{I(2I+1)} = \frac{3f+1}{3}$

11. $\{x + 1\} > x^2 - 2x$

$\{x\} > x^2 - 2x$

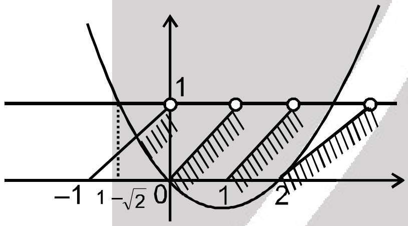
in $x \in (-1, 0)$

$\{x\} = x + 1$

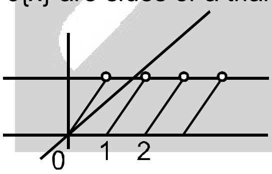
$\therefore \quad x + 1 > x^2 - 2x$

$x^2 - 3x - 1 < 0 \;\Rightarrow\; \left(x - \frac{3}{2}\right)^2 - \frac{13}{4} < 0 \;\Rightarrow\; x \in \left(\frac{3}{2} - \frac{\sqrt{13}}{2}, \frac{3}{2} + \frac{\sqrt{13}}{2}\right)$

Hence $x \in \left(\frac{3-\sqrt{13}}{2}, 2\right) - \{0\}$



12. $4x, 5[x], 6\{x\}$ are sides of a triangle



true for $x > 1$

$\therefore \quad x > 0, [x] > 0$

$\therefore \quad x \geq 1$

$x \notin I \quad$ hence $x > ?$

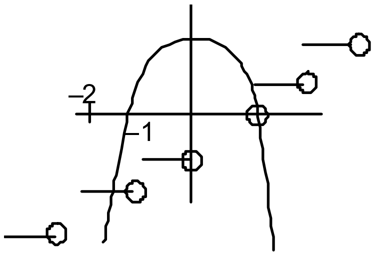
Case-1 $4x + 5[x] > 6\{x\}$	Case-2 $[x] + 6\{x\} > 4x$	Case-3 $4x + 6\{x\} > 5[x]$
$9x > 11\{x\}$	$5x + \{x\} > 4x$	$11\{x\} > x$
$\{x\} < \frac{9}{11}x$	for $x > 1$	$\{x\} > \frac{x}{11}$

$x \in \left(\frac{11}{10}, 2\right) \cup \left(\frac{22}{10}, 3\right) \cup \left(\frac{33}{10}, 4\right) \cup \left(\frac{44}{10}, 5\right) \cup \left(\frac{55}{10}, 6\right) \cup \ldots\ldots \cup \left(\frac{99}{10}, 10\right)$

13. **Case-I** $x\in(\sqrt{2},\sqrt{3}) \Rightarrow\ = x^2-2 \Rightarrow x^3-2x-1=0$

one root lies between $\sqrt{2}$ to $\sqrt{3}$ because $f(\sqrt{2}) = -1$ = negative& $f(\sqrt{3}) = \sqrt{3}-1$ = positive

14.



$x\in(-\infty,-\sqrt{3}\,]\cup[1,\infty)$

15. $[2x] = |x-1| + |x-3|$

case-I $x\in\left[0,\frac{1}{2}\right)$ $\quad 0 = 4-2x \quad\Rightarrow\quad x = 2$(rejected)

case-II $x\in\left[\frac{1}{2},1\right)$ $\quad 1 = 4-2x \quad\Rightarrow\quad x = \frac{3}{2}$ (rejected)

case-III $x\in\left[1,\frac{3}{2}\right)$ $\quad 2 = (x-1)+(3-x) \quad\Rightarrow\quad x\in\left[1,\frac{3}{2}\right)$

case-IV $x\in\left[\frac{3}{2},2\right)$ $\quad 3 = (x-1)+(3-x) \quad\Rightarrow\quad 2 = 3$ no solution

case-V $x\geq 2$ $\quad [2x] = (x-1)+(x-3)$

$\Rightarrow\quad 2x-4 = [2x] > 2x-1,$ hence no solution for $x\geq 2$

Hence set A is $\left[1,\frac{3}{2}\right)$ $\quad\Rightarrow\quad \alpha = \frac{3}{2}$

$$2|x-1| = \frac{[x]([x]-1)(2[x]-1)}{6} + [x]^2\{x\}$$

case-I $x\in[0,1)$ $\quad 2(1-x) = 0 \quad\Rightarrow\quad x = 1$(rejected)

case-II $x\in[1,2)$ $\quad 2x-2 = (x-1) \Rightarrow\quad x = 1$

case-III $\quad x\in[2,3)$ $\quad 2x-2 = \frac{2.1.3}{6} + 4(x-2)$

$\Rightarrow\quad 2x-2 = 1+4x-8 \quad\Rightarrow\quad x = \frac{5}{2}$

case-IV $\quad x\geq 3$

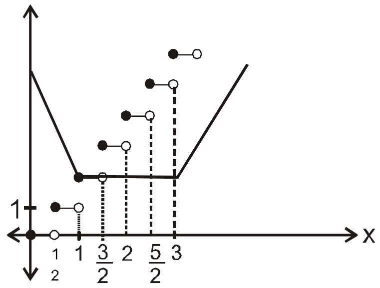
Now LHS is a linear expression whose slope is 2. RHS consists of segement spanning unit length along x whose slope is greater than or equal to 9. Morever at x = 3 LHS = 4 and RHS = 5, hence no solution for $x\geq 3$ as LHS < RHS in this interval.

Hence set B is $\left\{1,\frac{5}{2}\right\}$, so $\beta = 1$

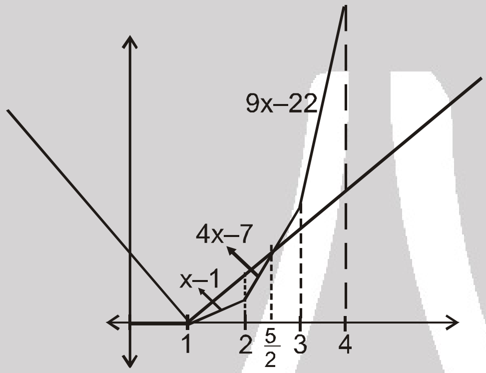
So $\quad A\cap B = \{1\}$, sum of elements of B = $\frac{7}{2}$

$$|2x-3|+|x-1| = \begin{cases} 4-3x & x\leq 1 \\ 2-x & 1<x\leq\frac{3}{2} \\ 3x-4 & x\geq\frac{3}{2} \end{cases} \quad\Rightarrow\quad M = \left(2-\frac{3}{2}\right) = \frac{1}{2}$$

Graphical Solution :



Hence A = $\left[1, \frac{3}{2}\right)$



Hence B = $\left\{1, \frac{5}{2}\right\}$

16. **Case -I** $x \in (-\infty, 0)$

$\Rightarrow \frac{x^2 - 5x - 6}{-x + 1} \geq 0 \Rightarrow \frac{(x+1)(x-6)}{x-1} \leq 0$

$\Rightarrow x \in (-\infty, -1] \cup (1, 6] \Rightarrow x \in (-\infty, -1]$

Case- II : $x = 0$ (which is sol)

Case- III : $x \in (0, 1)$

$\Rightarrow \frac{x^2 - 5x + 6}{-x + 1} \geq 0 \Rightarrow \frac{(x-2)(x-3)}{x-1} \leq 0$

$\Rightarrow x \in (-\infty, 1) \cup [2, 3] \Rightarrow x \in (0, 1)$

Case- IV : $x = 1$ (which is sol)

Case- V : $x \in (1, \infty)$

$\frac{x^2 - 5x + 6}{x + 1} \geq 0 \Rightarrow \frac{(x-2)(x-3)}{x+1} \geq 0$

$\Rightarrow$ $x \in (-1, 2] \cup [3, \infty)$

$\Rightarrow$ $x \in (1, 2] \cup [3, \infty)$

From all cases we get $x \in (-\infty, -1] \cup [0, 2] \cup [3, \infty)$

17. $f(x) = \sqrt{-\log_{\frac{x+4}{2}}\left(\log_2 \frac{2x-1}{3+x}\right)}$. For domain : $\log_{\frac{x+4}{2}}\left(\log_2 \frac{2x-1}{3+x}\right) \leq 0$

Case-I $0 < \frac{x+4}{2} < 1 \Rightarrow -4 < x < -2$ ....A

then $\log_{\frac{x+4}{2}}\left(\log_2 \frac{2x-1}{3+x}\right) \leq 0 \Rightarrow \log_2 \frac{2x-1}{3+x} \geq 1$

$\Rightarrow \quad \frac{2x-1}{3+x} \geq 2 \qquad \Rightarrow \quad x < -3$ ....B

$\Rightarrow$ on $A \cap B$ $\qquad x \in (-4, -3)$ ....(i)

Case-II $\quad \frac{x+4}{2} > 1 \quad$ or $\quad x > -2$ ....A

$\log_{\frac{x+4}{2}}\left(\log_2 \frac{2x-1}{3+x}\right) \leq 0 \Rightarrow \quad 0 < \log_2 \frac{2x-1}{3+x} \leq 1 \Rightarrow 1 < \frac{2x-1}{3+x} \leq 2$

$\Rightarrow \quad x \in (4, \infty)$ ....(ii)

$\therefore \quad$ (i) $\cup$ (ii) $\qquad$ Domain $x \in (-4, -3) \cup (4, \infty)$

18. $f(x) = (x^{12} - x^9 + x^4 - x + 1)^{-1/2}$

Dr : $x^{12} - x^9 + x^4 - x + 1 > 0$

For $x \leq 0$ it is obvious that for f(x) to be defined Dr > 0.

For $x \geq 1$, $(x^{12} - x^9) + (x^4 - x) + 1$ is positive

Since $x^{12} \geq x^9$, $x^4 \geq x$. For $0 < x < 1$, Dr = $x^{12} + (x^4 - x^9) + (1 - x) > 0$

Since $x^4 > x^9$, $x < 1$. Hence Dr > 0 for all $x \in R$

Domain is $x \in R$

19. $f(a) = \sqrt{2a^2 - a}$ for domain of f(x); $\qquad 2a^2 - a \geq 0 \quad \Rightarrow \quad a(2a-1) \geq 0$

$\therefore \quad a \in (-\infty, 0] \cup \left[\frac{1}{2}, \infty\right)$. Let $g(x) \equiv x^2 + (a+1)x + (a-1) = 0$

(i) $\quad D \geq 0$

$(a+1)^2 - 4(a-1) \geq 0 \quad \Rightarrow \quad a \in R$ ...(i)

(ii) $\quad -2 < -\frac{B}{2A} < 1 \quad \Rightarrow \quad -2 < -\frac{(a+1)}{2} < 1$

$\Rightarrow \quad a \in (-3, 3)$ ....(ii)

(iii) $\quad g(-2) > 0 \quad \Rightarrow \quad 4 - 2(a+1) + (a-1) > 0 \quad \Rightarrow \quad a < 1$

(iv) $\quad g(1) > 0 \Rightarrow \quad 1 + a + 1 + a - 1 > 0 \quad \Rightarrow \quad a > -1/2$

Now (i) $\cap$ (ii) $\cap$ (iii) $\cap$ (iv) we get $\qquad$ **Ans.** : $a \in \left(-\frac{1}{2}, 0\right] \cup \left[\frac{1}{2}, 1\right)$

20. $f(x) = \sqrt{\frac{1}{(|x|-1)\cos^{-1}(2x+1)\tan 3x}}$

here $\quad -1 \leq 2x + 1 < 1 \Rightarrow \quad -2 \leq 2x < 0 \quad \Rightarrow \quad -1 \leq x < 0 \quad \Rightarrow \quad x \in [-1, 0)$

But $x \neq -1$ $\quad$ as $\quad |x| - 1 \neq 0$

$\therefore \quad x \in (-1, 0)$

for $x \in (-1, 0)$, $(|x| - 1)$ is –ve

$\therefore \quad \tan 3x < 0$

$0 > 3x > -\frac{\pi}{2} \quad$ or $\quad x \in \left(-\frac{\pi}{6}, 0\right)$

Domain : $\left(-\frac{\pi}{6}, 0\right) \cap (-1, 0) \equiv \left(-\frac{\pi}{6}, 0\right)$

21. **(i)** $\sqrt{\log_{1/3}\log_4([x]^2-5)}$

Domain

(i) $\log_{1/3}\log_4([x]^2-5) \geq 0$ or $\log_4([x]^2-5) \leq 1$

or $[x]^2 \leq 9$ or $x \in [-3, 4)$ ....(i)

(ii) $\log_4([x]^2-5) > 0$

or $[x]^2 - 5 > 1$

or $x \in (-\infty, -2) \cup [3, \infty)$ ....(ii)

(iii) $[x]^2 - 5 > 0$

$x \in (-\infty, -2) \cup [3, \infty)$ ....(iii)

Now (i) $\cap$ (ii) $\cap$ (iii)

$\Rightarrow$ $x \in [-3, 2) \cup [3, 4)$

(ii) $f(x) = \dfrac{1}{[|x-1|]+[|12-x|]-11}$

Case-I $x > 12$

$f(x) = \dfrac{1}{[x]-1+[x]-12-11}$ $\Rightarrow$ $f(x) = \dfrac{1}{2([x]-12)}$

Now for f(x) to be defined $[x] \neq 12$ $\Rightarrow$ $x \notin [12, 13)$ but $x > 12$

Case-II $1 \leq x \leq 12$

$$f(x) = \frac{1}{[x]-1+12+[-x]-11} = \begin{cases} \dfrac{1}{[x]+(-1-[x])} & \text{if } x \notin I \\ \text{not defined} & \text{if } x \in I \end{cases} \Rightarrow x \notin \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

Case-III $x < 1$

$$f(x) = \frac{1}{1+[-x]+[-x]+12-11} \quad f(x) = \begin{cases} \dfrac{1}{2(1-[x])} & \text{if } x \in I \\ \dfrac{1}{-2[x]} & \text{if } x \notin I \end{cases} \Rightarrow x \notin (0,1) \quad \because x < 1$$

(iii) $f(x) = (x+0.5)^{\log_{(0.5+x)}\frac{x^2+2x-3}{4x^2-4x-3}}$

$x + 0.5 > 0,\ x + 0.5 \neq 1$ $\Rightarrow$ $x \in (-0.5, \infty)$ & $x \neq 0.5$ .....(A)

& $\dfrac{x^2+2x-3}{4x^2-4x-3} > 0$ or $\dfrac{(x+3)(x-1)}{(2x-3)(2x+1)} > 0$

or $x \in (-\infty, -3) \cup \left(-\dfrac{1}{2}, 1\right) \cup \left(\dfrac{3}{2}, \infty\right)$ .....(B)

(A) $\cap$ (B) $\therefore$ Domain of f(x) : $x \in \left(-\dfrac{1}{2}, 1\right) \cup \left(\dfrac{3}{2}, \infty\right) - \left\{\dfrac{1}{2}\right\}$

(iv) $f(x) = \dfrac{5}{\left[\dfrac{x-1}{2}\right]} - 3^{\sin^{-1}x^2} + \dfrac{(7x+1)!}{\sqrt{x+1}}$

$\left[\dfrac{x-1}{2}\right] \neq 0$ $\Rightarrow$ $x \notin [1, 3)$

& $x \in [-1, 1]$

& $x + 1 > 0$ $\Rightarrow$ $x \in (-1, \infty)$

& $7x + 1 \in w$

$\therefore$ Domain $\left\{-\dfrac{1}{7}, 0, \dfrac{1}{7}, \dfrac{2}{7}, \dfrac{3}{7}, \dfrac{4}{7}, \dfrac{5}{7}, \dfrac{6}{7}\right\}$

(v) $3^y = 2^{4x^2-1} - 2^{x^4} > 0$

$4x^2 - 1 > x^4 \Rightarrow (x^2)^2 - 4x^2 + 1 < 0$

$(x^2 - 2)^2 + 1 - 4 < 0 \qquad \sqrt{2-\sqrt{3}} < |x| < \sqrt{2+\sqrt{3}}$

$x \in \left(-\sqrt{2+\sqrt{3}}, -\sqrt{2-\sqrt{3}}\right) \cup \left(\sqrt{2-\sqrt{3}}, \sqrt{2+\sqrt{3}}\right)$

$x \in \left(\frac{-\sqrt{3}-1}{\sqrt{2}}, \frac{-\sqrt{3}+1}{\sqrt{2}}\right) \cup \left(\frac{\sqrt{3}-1}{\sqrt{2}}, \frac{\sqrt{3}+1}{\sqrt{2}}\right)$

22. $f(x) = \sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]$

Domain : $-1 \le \left[x^2 - \frac{1}{2}\right] \le 1 \Rightarrow x \in \left(-\sqrt{\frac{5}{2}}, \sqrt{\frac{5}{2}}\right)$

and $-1 \le \left[x^2 + \frac{1}{2}\right] \le 1 \Rightarrow x \in \left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)$

$\Rightarrow$ domain is $x \in \left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)$ or $x^2 \in \left[0, \frac{3}{2}\right)$

if (i) $x^2 \in \left[0, \frac{1}{2}\right)$, then $f(x) = \pi$

if (ii) $x^2 \in \left[\frac{1}{2}, 1\right)$, then $f(x) = \pi$

if (iii) $x^2 \in \left[1, \frac{3}{2}\right)$, then $f(x) = \pi \Rightarrow$ range = $\{\pi\}$

23. $f(x) = \frac{1}{2\{-x\}} + 1 - \{x\} - 1 = \frac{1}{2\{-x\}} + \{-x\} - 1 \ge \sqrt{2} - 1$ (Using A.M. ≥ G.M.) (A.M. ≥ G.M.)

Range of f(x) is $[\sqrt{2} - 1, \infty)$

24. $f(x) = \frac{\sqrt{x^2+1} - 3x}{\sqrt{x^2+1} + x} = \frac{1 - \frac{3x}{\sqrt{x^2+1}}}{1 + \frac{x}{\sqrt{x^2+1}}}$ Let $g(x) = \frac{x}{\sqrt{x^2+1}}$ and $h(x) = \frac{1-3x}{1+x}$. Thus $f(x) = h(g(x))$

Now, $g'(x) = \frac{\sqrt{x^2+1} - x\frac{1\,(2x)}{2\sqrt{x^2+1}}}{x^2+1} = \frac{1}{(x^2+1)^{3/2}} > 0$ and $h'(x) = \frac{(1+x)(-3) - (1-3x)(1)}{(1+x)^2} = \frac{-4}{(1+x)^2} < 0$

and $f'(x) = h'(g(x))\, g'(x) < 0$

$\Rightarrow$ minimum $f(x) = \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \frac{\sqrt{x^2+1} - 3x}{\sqrt{x^2+1} + x} = \lim\limits_{x\to\infty} \frac{\sqrt{1+\frac{1}{x^2}} - 3}{\sqrt{1+\frac{1}{x^2}} + 1} = -1$

$\Rightarrow$ maximum $h(x) = \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \frac{\sqrt{x^2+1} - 3x}{\sqrt{x^2+1} + x} = \lim\limits_{x\to-\infty} \frac{|x|\sqrt{1+\frac{1}{x^2}} - 3x}{|x|\sqrt{1+\frac{1}{x^2}} + x}$

$= \lim\limits_{x\to-\infty} \frac{-\sqrt{1+\frac{1}{x^2}} - 3}{-\sqrt{1+\frac{1}{x^2}} + 1} = \infty \Rightarrow$ Range of $f(x) = (-1, \infty)$

25. **(i)** $g(x) > 0 \Rightarrow |\sin x| + \sin x > 0 \Rightarrow 0 < x < \pi$ ....(A)

(ii) $0 < h(x) < 1$ or $h(x) > 1$

(a) $0 < \sin x + \cos x < 1 \Rightarrow \frac{\pi}{2} < x < \frac{3\pi}{4}$ ....(B)

(b) $\sin x + \cos x > 1 \Rightarrow 0 < x < \frac{\pi}{2}$ ....(C)

(iii) $\log_{h(x)} g(x) \geq 0$

since $h(x) > 1$, $g(x) \geq 1$

i.e. $|\sin x| + \sin x \geq 1 \Rightarrow \sin x \geq \frac{1}{2}$, $(\because \sin x > 0) \Rightarrow \frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$ ....(D)

From (C) and (D) $x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right)$

(b) $0 < h(x) < 1$ then $0 < g(x) \leq 1$

$0 < |\sin x| + \sin x \leq 1 \Rightarrow 0 < \sin x \leq \frac{1}{2}$

i.e. $0 < x \leq \frac{\pi}{6}$ & $\frac{5\pi}{6} \leq x < \pi$ .... (E)

From B & E $x \in \phi$ so final domain is $\left[\frac{\pi}{6}, \frac{\pi}{2}\right)$

26. **(i)** For domain (i) $[x] > 0$ and $[x] \neq 1$

so $[x] \geq 2$, so $x \in [2, \infty)$

for range if $x \in [2, \infty)$, then $\frac{|x|}{x} = 1$

so $f(x) = \cos^{-1} 0 = \frac{\pi}{2}$

Range of $f(x) = \left\{\frac{\pi}{2}\right\}$

(ii) $f(x) = \sqrt{\log_{1/2} \log_2 [x^2 + 4x + 5]}$

$D : 0 < \log_2 [x^2 + 4x + 5] \leq 1$

or $1 < [x^2 + 4x + 5] \leq 2$

$\Rightarrow [x^2 + 4x + 5] = 2$

or $2 \leq x^2 + 4x + 5 < 3$

$D : x \in (-2 - \sqrt{2}, -3] \cup [-1, -2 + \sqrt{2})$

$R : \{0\}$

(iii) $f(x) = \sin^{-1}\left[\log_2\left(\frac{x^2}{2}\right)\right] \Rightarrow -1 \leq \log_2\left(\frac{x^2}{2}\right) < 2 \Rightarrow \frac{1}{2} \leq \frac{x^2}{2} < 4$

$\Rightarrow x \in (-\sqrt{8}, -1] \cup [1, \sqrt{8})$ and $R : \left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$

(iv) $f(x) = \log_{[x-1]} \sin x$

$\sin x > 0 \Rightarrow x \in (2n\pi, (2n+1)\pi)$

here $[x-1] > 0$ & $[x-1] \neq 1 \Rightarrow x \in [3, \infty)$

Domain $x \in [3, \pi) \cup \left[\bigcup_{n=1}^{\infty} (2n\pi, (2n+1)\pi)\right]$.

For range $\sin x \in (0, 1]$ and $[x-1] \in [2, \infty)$ so range $\in (-\infty, 0]$

(v) $f(x) = \tan^{-1} \sqrt{[x]+[-x]} + \sqrt{2-|x|} + \frac{1}{x^2}$

Domain : (i) $[x] + [-x] \geq 0 \Rightarrow x \in I$

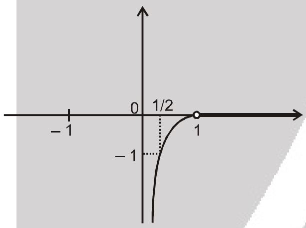
(ii) $2 - |x| \geq 0 \Rightarrow |x| \leq 2 \Rightarrow x \in [-2, 2]$

(iii) $x \neq 0$

For domain (i) $\cap$ (ii) $\cap$ (iii)

Domain : $\{-2, -1, 1, 2\}$

Range : $\left\{\frac{1}{4}, 2\right\}$



27. $y = \frac{\sin^2 x + 4\sin x + 4}{2\sin^2 x + 8\sin x + 8} + \frac{1}{2\sin^2 x + 8\sin x + 8} = \frac{1}{2} + \frac{1}{2(\sin x + 2)^2}$

$y_{max} = \frac{1}{2} + \frac{1}{2(-1+2)^2} = 1$; $y_{min} = \frac{1}{2} + \frac{1}{2(1+2)^2} = \frac{5}{9}$

$\therefore$ range $= \left[\frac{5}{9}, 1\right]$

28. $f(x) = \log_2 [3x - 3[x]] = \log_2 [3\{x\}]$

period 1.

so $0 \leq \{x\} < 1$

$0 \leq 3\{x\} < 3 \Rightarrow [3\{x\}] = 0, 1, 2$

so range $\{\log_2 1, \log_2 2\} = \{0, 1\}$.

29. As g(x) is periodic with period 2 so f(g(x)) is periodic with period 2.

Now $g(x) = \sin \pi x + g\left\{\frac{x}{2}\right\} = \sin \pi x + 8 \cdot \frac{x}{2} \cdot 0 \leq x < 2$

$g(x) = \sin \pi x + 4x$; $0 \leq x < 2 \Rightarrow g'(x) = \pi \cos \pi x + 4 \uparrow$.

so $g(0) = 0$ and $g(2^-) = 8$ so, $g(x) \in [0, 8)$

so range of f(g(x)) is $\left(\frac{1}{1+64}, \frac{1}{1}\right] \equiv \left(\frac{1}{64}, 1\right]$.

30. Period of $e^{\tan\left\{\frac{x}{4}\right\}}$ is 4 $\Rightarrow$ $\cos\pi\left(\frac{(1-2[x])}{2}\right) = 0 \quad \forall x \in R$

Period of $\sin\left(\frac{\pi[x]}{2}\right)$ is 4

$\therefore$ Period of f(x) is 4. For periodic function f(x) range can be calculated for $x \in [0, 4]$

If $x \in [0, 1)$; $f(x) = \frac{x}{4}$, $f(x) \in \left[0, \frac{1}{4}\right)$; If $x \in [1, 2)$; $f(x) = \frac{x}{4} + 1$, $f(x) \in \left[\frac{5}{4}, \frac{3}{2}\right)$

If $x \in [2, 3)$; $f(x) = \frac{x}{4}$, $f(x) \in \left[\frac{2}{4}, \frac{3}{4}\right)$; If $x \in [3, 4)$; $f(x) = \frac{x}{4} - 1$, $f(x) \in \left[-\frac{1}{4}, 0\right)$

$\therefore$ Range $\in \left[-\frac{1}{4}, \frac{1}{4}\right) \cup \left[\frac{2}{4}, \frac{3}{4}\right) \cup \left[\frac{5}{4}, \frac{3}{2}\right)$

31. $\sin\frac{\pi}{4}[x] + \cos\frac{\pi x}{2} + \cos\frac{\pi}{3}[x]$ period is LCM of 8, 4 and 6 = 24

32. After simplification $g(x) = \frac{x}{x-1}$ $\Rightarrow g(2) = 2$

33. $f(x) + 5 \le f(x + 5) \le f(x + 4) + 1 \le f(x + 3) + 2 \le f(x + 2) + 3 \le f(x + 1) + 4 \le f(x) + 5$

$\Rightarrow$ In all steps there is equality only $\Rightarrow$ $f(x + 1) = f(x) + 1$

Now $f(1) = 1$ $\Rightarrow$ $f(2) = 2$

$f(3) = 3$

$f(4) = 4$

$f(2016) = 2016$ $\Rightarrow$ $g(2016) = 2016 + 1 - 2016 = 1$

34. $[x]\,[y] = x + y$

(i) if $x, y \in I$ then $xy = x + y$ or $y = \frac{x}{x-1}$ $\Rightarrow$ (x, y) is (0, 0), (2, 2)

(ii) if $x, y \notin I$

Let $x = I_1 + f_1$ and $y = I_2 + f_2$ then $I_1 + I_2 + f_1 + f_2 = I_1I_2$

$\Rightarrow$ $f_1 + f_2 \in I$

$0 < f_1 + f_2 < 2$ $\Rightarrow$ $f_1 + f_2 = 1$.

$I_1 + I_2 + 1 = I_1I_2$ $\Rightarrow$ $I_1 = \frac{I_2+1}{I_2-1} = 1 + \frac{2}{I_2-1}$.

$I_2 - 1 = \pm 1, \pm 2,\quad I_2 = 2, 0, 3, -1$

$\therefore$ $I_1 = 3, -1, 2, 0$ $\Rightarrow$ $I_1 I_2 = 6, 0$

$x + y = I_1I_2$ $\Rightarrow$ $x + y = 0$ or $x + y = 6$

35. **(i)** $f(x) = Ax^2 + Bx + C$ $\Rightarrow$ $x \in I$ and $f(x) \in I$

at $x = 0$, $f(0) = C$ $\Rightarrow$ C is integer at $x = 1$, $f(1) = A + B + C$

$\because$ C is integer $\therefore$ A + B is also integer

at $x = -1$, $f(-1) = A - B + C$ $\Rightarrow$ $f(1) + f(-1) = 2A + 2C$

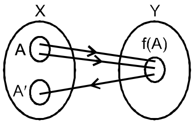
$\because$ C is integer $\therefore$ 2A is also integer

(ii) $f(x) = A\,x(x - 1) + (A + B)\,x + C$ $\Rightarrow$ $f(x) = 2A\,\frac{x(x-1)}{2} + (A + B)x + C$

If x is an integer then $\frac{x(x-1)}{2}$ is also an integer and $2A$, $\frac{x(x-1)}{2}$ $(A+ B)$, $C \in I$

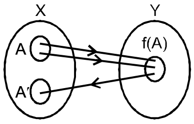
$\Rightarrow$ f(x) is also an integer.

36. For option A



$f^{-1}(f(A)) = A \cup A' \neq A$ Hence A is wrong

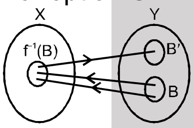
For option B



$f(X) = Y \Rightarrow$ f is onto but it will not effect on mapping of function. Hence B is wrong

For option A & B other explaination can be given else if Y is a singleton set then the function f is constant function and hence is trivially onto (unless $X = \phi$). But in such a case, even if A consists of just one point, f(A) is entire set Y and so $f^{-1}(f(A))$ is the entire set X, which could be much bigger than A. So A and B are wrong even if $f(X) = Y$

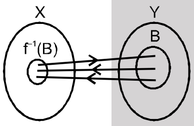
For option C



$f(X) \subset Y$ (range $\neq$ co-domain)

f(X) is a proper subset of Y (so that f is not onto), then for B = Y option C is wrong because $f^{-1}(Y) = X$ but $f(f^{-1}(Y)) = f(X) \neq Y$.

For option D



If B = Y, then $f(f^{-1}(Y))$ is the range of the function f. If this is equal to Y, then function must be onto, thus $f(X) = Y$ is necessary condition. Hence D is correct

37. For g(x) to be surjective $x \in R \Rightarrow 0 < \cos^{-1}\left(\dfrac{x^2 - k}{1 + x^2}\right) \leq \pi/3 \Rightarrow \dfrac{1}{2} \leq \dfrac{x^2 - k}{x^2 + 1} < 1$

$x^2 + 1 > 0 \quad \forall \quad x \in R$

$\dfrac{1}{2}(x^2 + 1) \leq x^2 - k < x^2 + 1$ ....(1)

From Eq. (1), taking RHS

$x^2 - k < x^2 + 1 \quad \Rightarrow \quad k > -1$

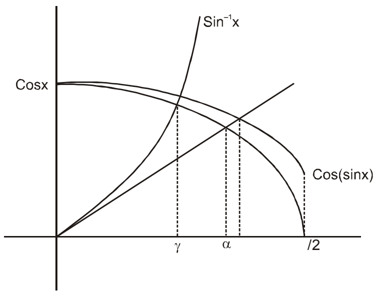
From Eq. (1), taking LHS

$x^2 + 1 \leq 2x^2 - 2k \quad \Rightarrow \quad x^2 \geq 2k + 1 \quad \forall \quad x \in R$

$2k + 1 \leq 0 \quad \Rightarrow \quad k \leq \dfrac{-1}{2} \quad \Rightarrow \quad k \in \left(-1, -\dfrac{1}{2}\right]$

38. Equation (1) is $\cos x = x$ equation (2) is $\cos(\sin x) = x$ and equation (3) is $\cos x = \sin^{-1}x$ is

Since $\sin x \leq x \Rightarrow \cos(\sin x) \geq \cos x$ now drawing the graphs of the functions we see that $\gamma < \alpha < \beta$



39. Given $f(x) = \log_2\log_3\log_4\log_5(\sin x + a^2)$

f(x) is defined only if $\log_3\log_4\log_5(\sin x + a^2) > 0,\ \forall\ x \in R$

$\Rightarrow \log_4\log_5(\sin x + a^2) > 1,\ \forall\ x \in R \quad \Rightarrow \quad \log_5(\sin x + a^2) > 4,\ \forall\ x \in R$

$\Rightarrow (\sin x + a^2) > 5^4,\ \forall\ x \in R \quad \Rightarrow \quad a^2 > 625 - \sin x,\ \forall\ x \in R$

$\Rightarrow a^2$ must be greater than maximum value of $625 - \sin x$ which is 626 (when $\sin x = -1$)

$\Rightarrow a^2 > 626 \quad \Rightarrow \quad a \in (-\infty, -\sqrt{626}) \cup (\sqrt{626}, \infty)$

40. **(i)** Let $\cos^{-1} x = \theta$, then $x = \cos\theta$ and $0 \le \theta \le \pi$

$$\therefore \quad \sin^{-1}\sqrt{1-x^2} = \sin^{-1}(\sin\theta) = \begin{cases} \theta & \text{if } 0 \le \theta \le \frac{\pi}{2} \\ \pi - \theta & \text{if } \frac{\pi}{2} < \theta \le \pi \end{cases} = \begin{cases} \cos^{-1}x & \text{if } 0 \le x \le 1 \\ \pi - \cos^{-1}x & \text{if } -1 \le x < 0 \end{cases}$$

$\therefore \quad \cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$ if $0 < x < 1$ is true.

(ii) Let $\sin^{-1} x = \theta$, then $x = \sin\theta$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\therefore \quad \cos^{-1}\sqrt{1-x^2} = \cos^{-1}(\cos\theta) = \begin{cases} -\theta & , -\frac{\pi}{2} \le \theta \le 0 \\ \theta & , 0 \le \theta \le \frac{\pi}{2} \end{cases} = \begin{cases} -\sin^{-1}x & , -1 \le x \le 0 \\ \sin^{-1}x & , 0 \le x \le 1 \end{cases}$$

$\therefore \quad \sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$ if $0 < x < 1$ is true

(iii) Let $\cos^{-1} x = \theta$, then $x = \cos\theta$ and $0 \le \theta \le \pi$

$$\therefore \quad \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \tan^{-1}(\tan\theta) = \begin{cases} \theta & , 0 \le \theta < \frac{\pi}{2} \\ \theta - \pi & , \frac{\pi}{2} < \theta \le \pi \end{cases} = \begin{cases} \cos^{-1}x & , 0 < x \le 1 \\ -\pi + \cos^{-1}x & , -1 \le x < 0 \end{cases}$$

i.e $\cos^{-1} x = \pi + \tan^{-1}\frac{\sqrt{1-x^2}}{x}$, $-1 < x < 0$ is correct.

41. Let $\operatorname{cosec}^{-1} x = \theta$, then $x = \operatorname{cosec}\theta$ and $\theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

$$\therefore \quad \cot(\operatorname{cosec}^{-1}x) = \cot\theta = \begin{cases} -\sqrt{\operatorname{cosec}^2\theta - 1} & \text{if } -\frac{\pi}{2} \le \theta < 0 \\ \sqrt{\operatorname{cosec}^2\theta - 1} & \text{if } 0 < \theta \le \frac{\pi}{2} \end{cases} = \begin{cases} -\sqrt{x^2-1} & \text{if } x \le -1 \\ \sqrt{x^2-1} & \text{if } x \ge 1 \end{cases}$$

42. **(i)** Let $\sin^{-1}x = \theta$. Then $x = \sin\theta$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\therefore \quad \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$$

$$\therefore \quad \cos^{-1}\sqrt{1-x^2} = \cos^{-1}(\cos\theta) = \begin{cases} -\theta & \text{if } -\frac{\pi}{2} \le \theta < 0 \\ \theta & \text{if } 0 \le \theta \le \frac{\pi}{2} \end{cases} = \begin{cases} -\sin^{-1}x & \text{if } -1 \le x < 0 \\ \sin^{-1}x & \text{if } 0 \le x \le 1 \end{cases}$$

$$\therefore \quad \sin^{-1}x = \begin{cases} -\cos^{-1}\sqrt{1-x^2} & \text{if } -1 \le x < 0 \\ \cos^{-1}\sqrt{1-x^2} & \text{if } 0 \le x \le 1 \end{cases}$$

(ii) Let $\sin^{-1}x = \theta$. Then $x = \sin\theta$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ {Note : $\theta \neq -\frac{\pi}{2}, \frac{\pi}{2}$ because $x \neq \pm 1$ }

$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{\sqrt{1-x^2}} \qquad \therefore \quad \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \tan^{-1}(\tan\theta) = \theta = \sin^{-1}x$

Thus $\sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$, for all $x \in (-1, 1)$

(iii) Let $\sin^{-1}x = \theta$. Then $x = \sin\theta$ and $-\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$ { Note : $\theta \neq 0$, because $x \neq 0$ }

$\therefore \quad \cot\theta = \frac{\sqrt{1-x^2}}{x}$

$$\therefore \quad \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}(\cot\theta) = \begin{cases} \theta+\pi & \text{if } -\frac{\pi}{2} \le \theta < 0 \\ \theta & \text{if } 0 < \theta \le \frac{\pi}{2} \end{cases} = \begin{cases} \pi + \sin^{-1}x & \text{if } -1 \le x < 0 \\ \sin^{-1}x & \text{if } 0 < x \le 1 \end{cases}$$

$$\text{Thus } \sin^{-1}x = \begin{cases} \cot^{-1}\frac{\sqrt{1-x^2}}{x} - \pi & \text{if } -1 \le x < 0 \\ \cot^{-1}\frac{\sqrt{1-x^2}}{x} & \text{if } 0 < x \le 1 \end{cases}$$

43. **Case I** $y = x$ $\quad x < 1$

$x = y$ $\quad y < 1$

$f^{-1}(x) = x$ $\quad x < 1$

Case II $y = x^2$ $\quad 1 \le x \le 4$

$x^2 = y$ $\quad 1 \le y \le 16$

$x = \sqrt{y}$ $\quad 1 \le y \le 16$

$f^{-1}(x) = \sqrt{x}$ $\quad 1 \le x \le 16$

Case III $y = 8\sqrt{x}$ $\quad x > 4$

$x = \frac{y^2}{64}$ $\quad y > 16$

$f^{-1}(x) = \frac{x^2}{64}$ $\quad x > 16$

44. $-1 \le \frac{x^2}{4} + \frac{y^2}{9} \le 1$ represents interior and the boundary of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ ....(i)

Also $-1 \le \frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} - 2 \le 1$

i.e. $\frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} \ge 1$ and $\frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} \le 3$

$\frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} \ge 1$ represents the portion of xy plane

which contains only one point viz : $\left(\sqrt{2}, \frac{3}{\sqrt{2}}\right)$ of $\frac{x^2}{4} + \frac{y^2}{9} \le 1$

$$\therefore \quad \sin^{-1}\left(\frac{x^2}{4} + \frac{y^2}{9}\right) + \cos^{-1}\left(\frac{x}{2\sqrt{2}} + \frac{y}{3\sqrt{2}} - 2\right) = \sin^{-1}\left(\frac{1}{2} + \frac{1}{2}\right) + \cos^{-1}\left(\frac{1}{2} + \frac{1}{2} - 2\right)$$

$$= \sin^{-1}1 + \cos^{-1}(-1) = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$$

45. $\alpha = 2\tan^{-1}\left(\dfrac{1-x}{1-x}\right)$; $\beta = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

put x = tan θ $\left\{\because\ x > 1 \Rightarrow \theta > \dfrac{\pi}{4}\right\}$

$\Rightarrow\quad \alpha = 2\tan^{-1}\left(\dfrac{1+\tan\theta}{1-\tan\theta}\right)$

$\Rightarrow\quad \alpha = 2\tan^{-1}\left\{\tan\left(\dfrac{\pi}{4}+\theta\right)\right\} = 2\left\{\dfrac{\pi}{4}+\theta-\pi\right\} = 2\left\{\theta-\dfrac{3\pi}{4}\right\} = 2\theta-\dfrac{3\pi}{2}$ ......(i)

$\beta = 2\sin^{-1}\left\{\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right\} = \sin^{-1}(\cos 2\theta) = \sin^{-1}\left[\sin\left(\dfrac{\pi}{2}-2\theta\right)\right] = \dfrac{\pi}{2}-2\theta$ ......(ii)

by (i) and (ii) $\alpha + \beta = -\pi$

46. Since $-1 \le x \le 1$

$\therefore\quad -\dfrac{\pi}{4} \le \tan^{-1}x \le \dfrac{\pi}{4}$

$\therefore\quad [\tan^{-1}x] = -1, 0$

When $[\tan^{-1}x] = -1$, then $\{\cos^{-1}x\} = 1$ (not possible)

When $[\tan^{-1}x] = 0$, then $\{\cos^{-1}x\} = 0$

$\therefore\quad \cos^{-1}x$ is integer

Since $0 \le \cos^{-1}x \le \pi$

$\therefore\quad \cos^{-1}x = 0, 1, 2, 3$

x = cos 0, cos 1, cos 2, cos 3

but $x \ne$ cos 2, cos 3

$\therefore$ the solution set is {1, cos 1}

47. If $x \le -1$, then $\sec^{-1}x > \dfrac{\pi}{2}$ and $\tan^{-1}x \le -\dfrac{\pi}{4} < 0$

$\therefore\quad \sec^{-1}x > \tan^{-1}x$ for all $x \le -1$

If $x \ge 1$, suppose $\tan^{-1}x = \theta$, then $\dfrac{\pi}{4} \le \theta < \dfrac{\pi}{2}$ and x = tan θ

$\therefore\quad \sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+x^2}$

$\therefore\quad \sec^{-1}\sqrt{1+x^2} = \sec^{-1}(\sec\theta) = \theta = \tan^{-1}x$

thus the inequality becomes $\sec^{-1}x > \sec^{-1}\sqrt{1+x^2}$

$\therefore\quad x > \sqrt{1+x^2}$ i.e. $x^2 > 1 + x^2$ which is not possible

$\therefore\quad \{x : x \in (-\infty, -1)\}$ is the solution set

48. $\sin^{-1}\sqrt{\dfrac{x}{1+x}} - \sin^{-1}\dfrac{x-1}{x+1} = \sin^{-1}\dfrac{1}{\sqrt{1+x}} \Rightarrow \sin^{-1}\sqrt{\dfrac{x}{1+x}} - \sin^{-1}\dfrac{1}{\sqrt{1+x}} = \sin^{-1}\dfrac{x-1}{x+1}$

$\Rightarrow\quad \sin^{-1}\left\{\sqrt{\dfrac{x}{1+x}}\sqrt{1-\dfrac{1}{1+x}} - \dfrac{1}{\sqrt{1+x}}\sqrt{1-\dfrac{x}{1+x}}\right\} = \sin^{-1}\left(\dfrac{x-1}{x+1}\right)$

$\Rightarrow\quad \sin^{-1}\left(\dfrac{x}{x+1}-\dfrac{1}{1+x}\right) = \sin^{-1}\left(\dfrac{x-1}{x+1}\right) \qquad x \in R$

But domain of $\sin^{-1}\sqrt{\dfrac{x}{1+x}} - \sin^{-1}\dfrac{x-1}{x+1} = \sin^{-1}\dfrac{1}{\sqrt{1+x}}$ is $x > 0$

Hence $x > 0$

49. (i) As $y \in I^+$ $\Rightarrow$ $\cot^{-1}y = \tan^{-1}\frac{1}{y}$ $\Rightarrow$ $\tan^{-1}\frac{1}{y} = \tan^{-1}\frac{3-x}{1+3x}$

$y = \frac{1+3x}{3-x}$ $\Rightarrow$ $x = 1, y = 2$ $\quad x = 2, y = 7$

(ii) case(i) x & y both are negative integers as $K \in N$ $\therefore$ RHS is +ve, while LHS is –ve

$\therefore$ no solution is possible

case(ii) x & y both are + ve integers.

As $x, y \in N$

$\therefore$ $\tan^{-1}x, \tan^{-1}y \in \{\tan^{-1}1, \tan^{-1}2, \tan^{-1}3........\}$

$\therefore$ $(\tan^{-1}x + \tan^{-1}y)$ has minimum possible value is $\frac{\pi}{2}$

But $\tan^{-1}k$ cannot be equal to $\frac{\pi}{2}$ or more

$\therefore$ No solution is possible

case(iii) One of them is +ve integer, while other is –ve integer say y is –ve I nteger.

Let $y = -p\ ;\ p \in N$

$\therefore$ Given equation $\Rightarrow$ $\tan^{-1}x - \tan^{-1}p = \tan^{-1}K$

$\Rightarrow$ $\tan^{-1}K + \tan^{-1}p = \tan^{-1}x$

Clearly no solution (similar to case (ii))

50. $\left.\begin{array}{l} 0 \le (\tan^{-1}x)^2 \le \frac{\pi^2}{4} \\ 0 \le (\cos^{-1}y)^2 \le \pi^2 \end{array}\right] \Rightarrow (\tan^{-1}x)^2 + (\cos^{-1}x)^2 \le \frac{5\pi^2}{4}$

But $(\tan^{-1}x)^2 + (\cos^{-1}x)^2 = \pi^2 k$ hence $k\pi^2 \le \frac{5\pi^2}{4}, k \le \frac{5}{4}$ ....(i)

Now put $\tan^{-1}x = \frac{\pi}{2} - \cos^{-1}y$ $\left(\frac{\pi}{2} - \cos^{-1}y\right)^2 + (\cos^{-1}y)^2 = \pi^2 k$ (where $\cos^{-1}y = t$)

$2t^2 - \pi t + \left(\frac{\pi^2}{4} - k\pi^2\right) = 0$

For real roots, $D \ge 0$

$\pi^2 - 8\left(\frac{\pi^2}{4} - k\pi^2\right) \ge 0$ $\Rightarrow$ $1 - 2 + 8k \ge 0, k \ge \frac{1}{8}$ ....(ii)

From (i) and (ii), k = 1

With k = 1, $t = \frac{\pi \pm \sqrt{8\pi^2 - \pi^2}}{4} = \frac{\pi + \sqrt{7}\pi}{4} = (1 \pm \sqrt{7})\frac{\pi}{4}$.

or $\cos^{-1}y = (\sqrt{7} + 1)\frac{\pi}{4}$ (as $0 \le \cos^{-1}y \le \pi$) $\therefore$ $y = \cos(\sqrt{7} + 1)\frac{\pi}{4}$

$\therefore$ $\tan^{-1}x = \frac{\pi}{2} - (\sqrt{7} + 1)\frac{\pi}{4} = \frac{\pi}{4}[(1 - \sqrt{7})]$ $\Rightarrow$ $x = \tan(1 - \sqrt{7})\frac{\pi}{4}$.